

# DIFFEOMORPHISM STABILITY AND CODIMENSION FOUR

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**ABSTRACT.** Given  $k \in \mathbb{R}$ ,  $v, D > 0$ , and  $n \in \mathbb{N}$ , let  $\{M_\alpha\}_{\alpha=1}^\infty$  be a Gromov-Hausdorff convergent sequence of Riemannian  $n$ -manifolds with sectional curvature  $\geq k$ , volume  $> v$ , and diameter  $\leq D$ . Perelman's Stability Theorem implies that all but finitely many of the  $M_\alpha$ s are homeomorphic. The Diffeomorphism Stability Question asks whether all but finitely many of the  $M_\alpha$ s are diffeomorphic.

We answer this question affirmatively in the special case when all of the singularities of the limit space occur along smoothly and isometrically embedded Riemannian manifolds of codimension  $\leq 4$ . We then describe several applications. For instance, if the limit space is an orbit space whose singular strata are of codimension at  $\leq 4$ , then all but finitely many of the  $M_\alpha$ s are diffeomorphic.

Let  $\mathcal{M}_{k,v,d}^{K,V,D}(n)$  denote the class of closed Riemannian  $n$ -manifolds  $M$  with

$$\begin{aligned} k &\leq \sec M \leq K, \\ v &\leq \text{vol } M \leq V, \quad \text{and} \\ d &\leq \text{diam } M \leq D, \end{aligned}$$

where  $\sec M$  is the sectional curvature of  $M$ ,  $\text{vol } M$  is the volume of  $M$ , and  $\text{diam } M$  is the diameter of  $M$ .

Let  $\{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  converge in the Gromov-Hausdorff topology to  $X$ . Perelman's Stability Theorem implies that all but finitely many of the  $M_\alpha$ s are homeomorphic to  $X$  ([20], [14]). Motivated by this it is natural to ask the

**Diffeomorphism Stability Question.** *Given  $k \in \mathbb{R}$ ,  $v, D > 0$ , and  $n \in \mathbb{N}$ , let  $\{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  be a convergent sequence. Are all but finitely many of the  $M_\alpha$ s diffeomorphic?*

If  $\{M_\alpha\}_{\alpha=1}^\infty$  happens to lie in  $\mathcal{M}_{k,v,0}^{K,\infty,D}(n)$  for some  $K \in \mathbb{R}$ , then by Gromov's Compactness Theorem,  $X$  is a  $C^{1,\alpha}$  Riemannian manifold, and all but finitely many of the  $M_\alpha$ s are  $C^1$ -diffeomorphic to  $X$  ([6], [18]).

An affirmative answer to the Diffeomorphism Stability Question would provide a simultaneous generalization of the Finiteness Theorems of Cheeger ([3]) and Grove-Petersen-Wu ([8]). In addition, Grove and the second author proved the following.

**Theorem.** ([11]) *If the answer to the Diffeomorphism Stability Question is "yes", then every Riemannian  $n$ -manifold  $M$  with  $\sec M \geq 1$  and  $\text{diam } M > \frac{\pi}{2}$  is diffeomorphic to  $S^n$ .*

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We answer the Diffeomorphism Stability Question affirmatively in the special case when all the singularities of  $X$  occur along smoothly and isometrically embedded Riemannian manifolds of codimension  $\leq 4$ . Before stating the result, we define the concept of a space being diffeomorphically stable.

**Definition A.** A space  $X \in \text{closure} \left( \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n) \right)$  is diffeomorphically stable if for any sequence  $\{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  with  $M_\alpha \rightarrow X$ , in the Gromov–Hausdorff topology, all but finitely many of the  $M_\alpha$ s are diffeomorphic.

The notion of a non-singular point we use was introduced in [1] where it is called a “ $(n, \delta)$ –burst point”. Elsewhere in the literature,  $(n, \delta)$ –burst points are called  $(n, \delta)$ –strained points (see also Definition 1.2, below).

**Theorem B.** There is a  $\delta(k, v, D, n) > 0$  so that  $X \in \text{closure} \left( \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n) \right)$  is diffeomorphically stable provided  $X$  contains a finite collection  $\mathcal{S} \equiv \{S_i\}_{i \in I}$  of smoothly and isometrically embedded, pairwise disjoint, Riemannian manifolds  $S_i$  without boundary that have the following properties.

1. Every point of  $X \setminus \{\cup_{i \in I} S_i\}$  is  $(n, \delta)$ –strained.
2. No point of any  $S \in \mathcal{S}$  is  $(\dim(S) + 1, \delta)$ –strained.
3.  $\mathcal{S}$  is the union of two subcollections  $\mathcal{K}$  and  $\mathcal{N}$ .
4. Elements of  $\mathcal{K}$  are compact and have codimension  $\leq 4$ .
5. Elements of  $\mathcal{N}$  are not compact and have codimension  $\leq 3$ .
6. The closure of an element of  $\mathcal{N} \in \mathcal{N}$  is a union of elements of  $\mathcal{S}$ .

Adopting the language of orbit spaces, we call the elements of  $\mathcal{S}$  the “strata” of  $X$ , and we call  $X \setminus \{\cup_{i \in I} S_i\}$  “the top strata”. It was shown in [1] that for all sufficiently small  $\delta > 0$ , the set,  $X_{n,\delta}$ , of  $(n, \delta)$ –strained points is a topological manifold that is open and dense in  $X$ . In general,  $X \setminus X_{n,\delta}$  can be rather wild, so the hypothesis that the singularities occur along Riemannian manifolds is rather special. Nevertheless this special situation occurs in all orbit spaces, so Theorem B has the following corollary.

**Corollary C.** If  $X \in \text{closure} \left( \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n) \right)$  is the quotient of an isometric group action on a Riemannian manifold, then  $X$  is diffeomorphically stable provided all of its singular strata have codimension  $\leq 4$ .

Theorem B generalizes Theorem 6.1 in [15], where the same conclusion is obtained under the hypothesis that  $\mathcal{S} = \emptyset$ . Theorem B also provides an alternative proof of the main theorems in [25] and [26]. The first author has observed that another consequence of Theorem B is that Theorem 1 in [24] holds with “homeomorphic” replaced with “diffeomorphic”. In other words, the following is a corollary of Theorem B and Theorem 1 in [24].

**Theorem D.** Let  $\mathcal{S}_k^n$  be the complete, simply connected Riemannian manifold with constant curvature  $k$ .

Given  $k, h, r \in \mathbb{R}$  and  $n \in \mathbb{N}$  with  $h, r \in (0, \frac{1}{2} \text{diam} \mathcal{S}_k^n]$  and  $h \leq r$ , there is an integer  $c$  with the following property.

If  $M$  is a complete Riemannian  $n$ -manifold with

$$\begin{aligned} \sec M &\geq k, \\ \text{Radius}(M) &\leq r, \\ \text{Sag}_r(M) &\leq h, \end{aligned} \tag{0.0.1}$$

and almost maximal volume, then  $M$  is diffeomorphic either to  $\mathbb{S}^n$ , to  $\mathbb{R}P^n$ , or to a Lens space  $\mathbb{S}^n/\mathbb{Z}_m$  with  $m \in \{3, 4, \dots, c\}$ .

We refer the reader to [24] for the definition of  $\text{Sag}_r(M)$  and the meaning of almost maximal volume with respect to the bounds in (0.0.1).

For the purpose of Theorem B, we use the following definition of smooth and isometric.

**Definition E.** Let  $X$  be an Alexandrov space and  $(S, g)$  a Riemannian manifold. Let  $\text{dist}^X$  be the distance of  $X$  and  $\text{dist}^S$  the distance on  $S$  induced by  $g$ . An embedding  $\iota : (S, g) \hookrightarrow X$  is smooth and isometric if the following hold.

1. There is a neighborhood  $U$  of the diagonal,  $\Delta(S) \subset S \times S$ , so that  $\iota^* \text{dist}^X$  is smooth on  $U \setminus \Delta(S)$ .
2. For every  $\varepsilon > 0$ , there is  $\delta > 0$  so that

$$|D_V \iota^* (\text{dist}^X(\cdot, \cdot)) - D_V \text{dist}^S(\cdot, \cdot)| < \varepsilon \tag{0.0.2}$$

for all unit  $V \in T(B(\Delta(S), \delta) \setminus \Delta(S))$ . Here  $D_V$  is the directional derivative operator associated to  $V$ , and  $B(\Delta(S), \delta)$  is the  $\delta$ -neighborhood of the diagonal in  $S \times S$  with respect to  $\text{dist}^S$ .

From here on we identify  $S$  with  $\iota(S)$  and write  $\text{dist}^X(\cdot, \cdot)$  for  $\iota^* \text{dist}^X$ .

**Remark.** When  $S$  is totally geodesic in  $X$ , then of course  $D_V \text{dist}^X(\cdot, \cdot)$  and  $D_V \text{dist}^S(\cdot, \cdot)$  coincide at points that are close enough to  $\Delta(S)$ .

When  $X$  is a smooth Riemannian manifold  $M$  and  $\iota$  is smooth, one can show that the condition in Definition E is equivalent to the embedding  $S \hookrightarrow M$  being Riemannian. In general, it implies that at points of  $S$  the space of directions of  $X$  contains a euclidean unit sphere of dimension  $\dim(S) - 1$ . (See Proposition 2.7 below.) It also implies that the intrinsic metrics on  $S$  induced by  $\text{dist}^X$  and  $g$  coincide, though the converse is false. For example, the boundary of a square with the intrinsic metric induced from  $\mathbb{R}^2$  does not satisfy (0.0.2). On the other hand, if  $X$  is the  $n$ -dimensional cube  $[0, 1]^n$ , then the open faces of  $X$  are smoothly and isometrically embedded submanifolds.

Here are some examples that illustrate the smoothness condition and the possibilities for the strata inclusions in Theorem B.

**Examples.** Let  $D^n$  be a disk in  $\mathbb{R}^n$  with boundary  $S^{n-1}$  and interior  $B^n$ . The double of  $D^{n-q} \times D^p \times D^r$  satisfies the hypotheses of Theorem B with

$$\begin{aligned} \mathcal{N} &= \{S^{n-q-1} \times S^{p-1} \times B^r, S^{n-q-1} \times B^p \times S^{r-1}, B^{n-q} \times S^{p-1} \times S^{r-1}\} \text{ and} \\ \mathcal{K} &= \{S^{n-q-1} \times S^{p-1} \times S^{r-1}\}. \end{aligned}$$

Thus the double of  $D^{n-q} \times D^p \times D^r$  is diffeomorphically stable. For similar reasons, the doubles of

$$D^{n-q} \times D^q \text{ and } D^{n-q} \times D^p \times D^r \times D^s$$

are diffeomorphically stable. More generally, in all the above examples, we may replace any of the disks  $D$  by any closed, convex subset  $C$  of any Riemannian manifold, provided the boundary of  $C$  is smooth and  $\dim(C) = \dim(D)$ .

To explain the strategy of the proof of Theorem B, let  $\{M_\alpha\}_\alpha \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  converge to  $X$ , and let  $G$  be a precompact open subset of the top stratum,  $X \setminus \{\cup_{i \in I} S_i\}$ . It follows from Theorem 6.1 in [15] that for all sufficiently large  $\alpha, \beta$ , there is an open  $G_\alpha \subset M_\alpha$  that is close to  $G$  and admits a smooth embedding  $\Phi_{\beta,\alpha} : G_\alpha \rightarrow M_\beta$  that is also a Gromov-Hausdorff approximation. The goal is to reconstruct  $\Phi_{\beta,\alpha}$  in a manner that extends to a diffeomorphism  $M_\alpha \rightarrow M_\beta$ . The next two results are the main tools that allow us to do this. The first is a consequence of the fact that the diffeomorphism group of the  $n$ -sphere deformation retracts to the orthogonal group when  $n = 1, 2$ , or  $3$  (see [12], [28]).

**Lemma F.** (*Bundle Extension Lemma*) *Let  $\pi_1 : E_1 \rightarrow B$  and  $\pi_2 : E_2 \rightarrow B$  be vector bundles. Let  $\pi_1 : A(E_1) \rightarrow B$  and  $\pi_2 : A(E_2) \rightarrow B$  be annulus bundles obtained by removing open unit disk bundles from  $\pi_1 : E_1 \rightarrow B$  and  $\pi_2 : E_2 \rightarrow B$  where the unit disk bundles are defined via euclidean metrics on  $E_1$  and  $E_2$ .*

*If*

$$\Phi : A(E_1) \rightarrow A(E_2)$$

*is a diffeomorphism so that*

$$\pi_1 = \pi_2 \circ \Phi, \tag{0.0.3}$$

*then  $\Phi$  extends to a diffeomorphism*

$$\hat{\Phi} : E_1 \rightarrow E_2$$

*so that  $\pi_1 = \pi_2 \circ \hat{\Phi}$ , provided the fiber dimension is  $\leq 4$ .*

We omit the proof of the Bundle Extension Lemma as it is very similar to Lemma 3.18 in [10].

There are two main difficulties with the proposal to extend  $\Phi_{\beta,\alpha}$  over successively lower dimensional strata: We do not have any canonical tubular neighborhoods around the strata to serve as the vector bundles of the Bundle Extension Lemma, and, even granting the existence of these vector bundles, we do not know that  $\Phi_{\beta,\alpha}$  satisfies (0.0.3).

We resolve these problems via the next result, which is the main new tool in the proof of Theorem B.

**Tubular Neighborhood Stability Theorem.** *Let  $X, \mathcal{S} \equiv \{S_i\}_{i \in I}, \mathcal{K}$ , and  $\mathcal{N}$  be as in Theorem B. Let  $\{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  converge to  $X$ . For all but finitely many  $\gamma \in \mathbb{N}$ ,  $M_\gamma$  has a finite open cover  $\{G_\gamma, \mathcal{U}_\gamma^{S_i}\}_{i \in I}$  with the following properties.*

1. *For  $i \neq j$ ,  $\mathcal{U}_\gamma^{S_i} \cap \mathcal{U}_\gamma^{S_j} = \emptyset$  unless  $S_i \subset \bar{S}_j$  or  $S_j \subset \bar{S}_i$ .*
2. *There are  $C^1$ -vector bundles*

$$P_\gamma^{S_i} : \mathcal{U}_\gamma^{S_i} \rightarrow O_{S_i} \subset S_i$$

*with  $O_{S_i} = P_\gamma^{S_i}(\mathcal{U}_\gamma^{S_i})$  an open subset of  $S_i$ . Moreover, if  $S_i \in \mathcal{K}$ , then  $O_{S_i} = S_i$ .*

3. *There are euclidean metrics on the  $\mathcal{U}_\gamma^{S_i}$ s so that  $G_\gamma = M_\gamma \setminus \overline{\cup_{i \in I} \mathcal{U}_\gamma^{S_i}(1)}$ , where  $\mathcal{U}_\gamma^{S_i}(t)$  is the bundle of open disks of radius  $t$  inside of  $\mathcal{U}_\gamma^{S_i}$ .*

4. For all but finitely many  $\alpha, \beta \in \mathbb{N}$ , there is a  $C^1$ -diffeomorphism

$$\Phi_{\beta, \alpha} : G_\alpha \longrightarrow \Phi_{\beta, \alpha}(G_\alpha) \subset G_\beta$$

so that for all  $S_i \in \mathcal{S}$

$$P_\alpha^{S_i} = P_\beta^{S_i} \circ \Phi_{\beta, \alpha}, \quad (0.0.4)$$

wherever both expressions are defined.

5. Given  $N_k \in \mathcal{N}$ , let

$$I_k \equiv \{j \in I \mid S_j \subset \bar{N}_k \setminus N_k\}.$$

For each  $j \in I_k$ , there is a neighborhood  $\mathcal{V}^{S_j}$  of  $S_j$  in  $\bar{N}_k$  and a  $C^1$ -submersion

$$Q^{S_j} : \mathcal{V}^{S_j} \setminus S_j \longrightarrow S_j$$

so that

$$P_\gamma^{S_j} = Q^{S_j} \circ P_\gamma^{N_k}, \quad (0.0.5)$$

wherever both expressions are defined. Moreover,  $\{O_{N_k}, \mathcal{V}^{S_j}\}_{j \in I_k}$  covers  $\bar{N}_k$ .

6. The diffeomorphisms  $\Phi_{\beta, \alpha}$  from Part 4 satisfy

$$\Phi_{\beta, \alpha}(M_\alpha \setminus \cup_i \mathcal{U}_\alpha^{S_i}(3)) = M_\beta \setminus \cup_i \mathcal{U}_\beta^{S_i}(3).$$

Since we will frequently refer to the Tubular Neighborhood Stability Theorem for brevity we will call it the TNST.

To prove Theorem B, we successively extend the diffeomorphism  $\Phi_{\beta, \alpha} : G_\alpha \longrightarrow \Phi_{\beta, \alpha}(G_\alpha)$  of the TNST to the lower dimensional strata. This is done by combining Equations (0.0.4) and (0.0.5) with the Bundle Extension Lemma. In order to apply the Bundle Extension Lemma, we must know that the complement of  $\Phi_{\beta, \alpha}(G_\alpha)$  within each  $\mathcal{U}_\beta^{S_i}$  is a disk bundle. To achieve this, we need one further ingredient that we call the (Step *a*)-Schoenflies Lemma.

Before stating the (Step *a*)-Schoenflies Lemma, we define the concept of the *Ancestor Number* of a stratum  $S$ . It is related to the concept of Descendent Number that appears in [27]. Set  $\mathcal{S}^{\text{ext}} \equiv \mathcal{S} \cup (X \setminus \cup_{S \in \mathcal{S}} S)$ , and partially order the  $S \in \mathcal{S}^{\text{ext}}$  by declaring that  $S < S'$  if  $S \subsetneq \bar{S}'$ , where  $\bar{S}'$  is the closure of  $S'$ . We call  $a \in \mathbb{N}$  the Ancestor Number of  $S \in \mathcal{S}^{\text{ext}}$  if  $a$  is the length of the largest chain

$$S_a < \cdots < S_1 < S_0$$

with  $S = S_a$  and  $S_0 = X \setminus \{\cup_{i \in I} S_i\}$ .

**Example.** Let  $X$  be the double of the 5-dimensional cube  $([0, 1]^5)_- \amalg_\partial ([0, 1]^5)_+$ . Then the strata and their Ancestor Numbers are given by the following table.

Submanifold	Ancestor Number
Interiors of the cubes and their 4-dimensional faces	0
Interiors of the 3-dimensional faces	1
Interiors of the 2-dimensional faces	2
Interiors of the 1-dimensional faces	3
Vertices	4

The fact that the 4-dimensional faces in this example are part of the top stratum illustrates a more general phenomenon: Since  $\partial X = \emptyset$ , it follows from Corollary 12.8 of [1] that the  $(n-1)$ -strained points of  $X$  are  $n$ -strained. If  $X$  is as in Theorem B, then all of the  $S_i$ s are of codimension  $\leq 4$ . It follows that the only possible Ancestor Numbers are 0, 1, 2, and 3.

Let  $\mathfrak{U}_\gamma^a(r)$  be the union of all  $\mathcal{U}_\gamma^{S_i}(r)$ s for which the Ancestor Number of  $S_i$  is  $a$ .

To prove Theorem B, assume that  $M_\alpha, M_\beta \in \mathcal{M}_{k,v,0}^{\infty,\infty,D}(n)$  are sufficiently close to  $X$ . For  $a = 0, 1, 2$ , or 3, we outline how to construct  $C^1$ -embeddings

$$\Phi_{\beta,\alpha}^a : M_\alpha \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\alpha^i(3)\} \longrightarrow M_\beta$$

by induction on the Ancestor Number,  $a$ . The map  $\Phi_{\beta,\alpha}^3$  will then be our desired diffeomorphism between  $M_\alpha$  and  $M_\beta$ .

The top stratum has Ancestor Number 0. The diffeomorphism  $\Phi_{\beta,\alpha} : G_\alpha \longrightarrow \Phi_{\beta,\alpha}(G_\alpha) \subset G_\beta$  of Part 4 of the TNST anchors the induction at Step 0.

To explain the induction step, suppose that we have constructed a smooth embedding

$$\Phi_{\beta,\alpha}^a : M_\alpha \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\alpha^i(3)\} \longrightarrow M_\beta$$

so that

$$P_\alpha^{S_k} = P_\beta^{S_k} \circ \Phi_{\beta,\alpha}^a, \quad (0.0.6)$$

wherever both expressions are defined, and so that the following lemma is satisfied.

**(Step  $a$ )–Schoenflies Lemma:** The image of  $M_\alpha \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\alpha^i(3)\}$  under  $\Phi_{\beta,\alpha}^a$  is  $M_\beta \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\beta^i(3)\}$ , that is,

$$\Phi_{\beta,\alpha}^a(M_\alpha \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\alpha^i(3)\}) = M_\beta \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\beta^i(3)\}.$$

Equation (0.0.6) gives us Equation (0.0.3) with  $P_\alpha^{S_k}$ ,  $P_\beta^{S_k}$ , and  $\Phi_{\beta,\alpha}^a$  playing the roles of  $\pi_1$ ,  $\pi_2$ , and  $\Phi$ , respectively. By the (Step  $a$ )–Schoenflies Lemma  $\Phi_{\beta,\alpha}^a$  restricts to an embedding of an annulus subbundle of  $P_\alpha^{S_k}$  to an annulus subbundle of  $P_\beta^{S_k}$ . Thus combining Equation (0.0.6) with the Bundle Extension Lemma, we extend  $\Phi_{\beta,\alpha}^a$  to a smooth embedding

$$\Phi_{\beta,\alpha}^{a+1} : M_\alpha \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\alpha^i(3)\} \longrightarrow M_\beta$$

that satisfies

$$P_\alpha^{S_k} = P_\beta^{S_k} \circ \Phi_{\beta,\alpha}^{a+1}, \quad (0.0.7)$$

wherever both expressions are defined and provided the Ancestor Number of  $S_k$  is  $a+1$ .

To check that Equation (0.0.7) holds when the Ancestor Number of  $S_k$  is  $\geq a+2$ , suppose  $S_k \subset \bar{S}_l$  and the Ancestor Number of  $S_l$  is  $a+1$ . Apply  $Q^{S_k}$  to both sides of Equation (0.0.7) and use Part 5 of the TNST to get

$$P_\alpha^{S_k} = Q^{S_k} \circ P_\alpha^{S_l} = Q^{S_k} \circ P_\beta^{S_l} \circ \Phi_{\beta,\alpha}^{a+1} = P_\beta^{S_k} \circ \Phi_{\beta,\alpha}^{a+1}. \quad (0.0.8)$$

This completes the induction step modulo establishing the (Step  $a+1$ )–Schoenflies Lemma. So to prove Theorem B, it remains to establish the TNST and the (Step 0, 1, and 2)–Schoenflies Lemmas.

We prove the (Step 0, 1, and 2)–Schoenflies Lemmas by 3-step induction on the Ancestor Number  $a$ . The (Step 0)–Schoenflies Lemma is a consequence of Parts 4 and 6 of the TNST. To establish the lemma in general, we will construct vector fields (see Proposition 7.2) that

are related to the vector bundles of the TNST, which we will then use to complete the induction step.

The table below lists the main milestones in the remainder of the paper and in their roles in the overall proof.

<b>Lemma 2.11</b>	constructs local versions of the vector fields we use to prove the (Step a)–Schoenflies Lemma
<b>Theorem 2.14</b>	constructs a cover of $X$ by strained neighborhoods on which local Alexandrov models of the vector bundles of the TNST are defined
<b>Theorem 3.4</b>	constructs local approximate versions of the $P_\gamma^{S_i}$ s, $Q^{S_j}$ s, and $\Phi_{\alpha,\beta}$ s that satisfy local versions of Inequalities (0.0.4) and (0.0.5)
<b>Propositions 4.2 and 4.3</b>	show that the local maps from Theorem 3.4 are $C^1$ –close on their overlaps
<b>Corollary 5.5</b>	allows us to define the $P_\gamma^{S_i}$ s, $Q^{S_j}$ s, and $\Phi_{\alpha,\beta}$ s by gluing together the local approximate versions of the $P_\gamma^{S_i}$ s, $Q^{S_j}$ s, and $\Phi_{\alpha,\beta}$ s in a manner that preserves Inequalities (0.0.4) and (0.0.5)
<b>Proposition 7.2</b>	constructs the vector fields we use to prove the (Step a)–Schoenflies Lemma

The first subsection of Section 1 reviews basic concepts of Alexandrov geometry, and the second subsection uses these to derive several results that we use to prove the TNST. The bulk of the paper, Sections 2–6 and 8, is devoted to proving the TNST. The main project is the construction of the vector bundles of Part 2 of the TNST. To do this we glue together locally defined vector bundles whose projection mappings are  $C^1$ –close. We obtain Alexandrov models of these local vector bundles in Section 2, wherein we study isometric embeddings of Riemannian manifolds in Alexandrov spaces in greater detail. In Section 3, we construct the local vector bundles by combining strainers with Perelman’s concavity construction. Section 4 shows that the locally defined submersions from Section 3 are  $C^1$ –close.

The gluing result we use, Corollary 5.5, is stated in Section 5. Since it is similar to other results in the literature, we defer its proof to Appendix A (Section 8). We complete the proof of the TNST in Section 6 and establish the (Step 0, 1, and 2)–Schoenflies Lemmas in Section 7.

For the convenience of the reader, we list notations and conventions in Appendix B (Section 9).

**Remark.** *With no modifications of our proof of Theorem B, the hypothesis that the singular set is  $\cup \mathcal{S}$  can be replaced by the assumption that it is contained in  $\cup \mathcal{S}$  and every point of  $\cup \mathcal{N}$  is singular.*

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## 1. BASIC TOOLS OF ALEXANDROV GEOMETRY

The notion, from [1], of strainers in an Alexandrov space forms the core of the calculus arguments we use. In the next subsection, we review this notion and its relevant consequences. The exposition borrows freely from [25] and [26].

### 1.1. Strainers and their Consequences.

**Definition 1.2.** *Let  $X$  be an Alexandrov space. A point  $x \in X$  is said to be  $(n, \delta, r)$ -strained by the strainer  $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$  provided that for all  $i \neq j$  we have*

$$\begin{aligned} \tilde{\angle}(a_i, x, b_i) &> \pi - \delta, \quad \tilde{\angle}(a_i, x, b_j) > \frac{\pi}{2} - \delta, \\ \tilde{\angle}(b_i, x, b_j) &> \frac{\pi}{2} - \delta, \quad \tilde{\angle}(a_i, x, a_j) > \frac{\pi}{2} - \delta, \quad \text{and} \\ \min_{i=1, \dots, n} \{\text{dist}(\{a_i, b_i\}, x)\} &> r. \end{aligned}$$

*We say  $B \subset X$  is an  $(n, \delta, r)$ -strained set with strainer  $\{a_i, b_i\}_{i=1}^n$  provided every point  $x \in B$  is  $(n, \delta, r)$ -strained by  $\{a_i, b_i\}_{i=1}^n$ . When there is no need to specify,  $r$  we say that  $x$  is  $(n, \delta)$ -strained.*

Next we state a powerful lemma from [1] which shows that for a  $(1, \delta, r)$  strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

**Lemma 1.3.** ([1], Lemma 5.6) *Let  $B \subset X$  be  $(1, \delta, r)$ -strained by  $(a, b)$ . For any  $x, z \in B$ ,*

$$\begin{aligned} |\angle(a, x, z) - \tilde{\angle}(a, x, z)| &< \tau(\delta) + \tau(\text{dist}(x, z) | r), \quad \text{and} \\ |\angle(b, x, z) - \tilde{\angle}(b, x, z)| &< \tau(\delta) + \tau(\text{dist}(x, z) | r). \end{aligned} \tag{1.3.1}$$

*In addition,*

$$|\tilde{\angle}(a, x, z) + \tilde{\angle}(b, x, z) - \pi| < \tau(\delta) + \tau(\text{dist}(x, z) | r). \tag{1.3.2}$$

The importance of the previous result cannot be overstated. As we will see next, Lemma 5.6 of [1] gives us two-sided bounds for both the angle and the comparison angle of a strained point to its strainer. The tremendous synergy this creates is due to the fact that comparison angles are continuous and angles determine derivatives of distance functions.

**Lemma 1.4.** *Let  $B \subset X$  be  $(l, \delta, r)$ -strained by  $\{(a_i, b_i)\}_{i=1}^l$ . For any  $x \in B$  and  $i \neq j$ ,*

$$\begin{aligned} \pi - \delta &< \tilde{\angle}(a_i, x, b_i) \leq \pi, & \frac{\pi}{2} - \delta &< \tilde{\angle}(a_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), \\ \frac{\pi}{2} - \delta &< \tilde{\angle}(b_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), & \frac{\pi}{2} - \delta &< \tilde{\angle}(a_i, x, a_j) < \frac{\pi}{2} + \tau(\delta), \\ \pi - \delta &< \angle(a_i, x, b_i) \leq \pi, & \frac{\pi}{2} - \delta &< \angle(a_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), \quad \text{and} \\ \frac{\pi}{2} - \delta &< \angle(b_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), & \frac{\pi}{2} - \delta &< \angle(a_i, x, a_j) < \frac{\pi}{2} + \tau(\delta). \end{aligned}$$



*Proof.* Since angles are bigger than comparison angles, it follows from the definition of strainer that we need only prove the last three angle upper bounds.

Since angles are limits of comparison angles, our lower curvature bound gives us that

$$\angle(a_i, x, b_i) + \angle(b_i, x, b_j) + \angle(b_j, x, a_i) \leq 2\pi$$

(see [1], 2.3(D)). Since angles are bigger than comparison angles, the definition of strainer gives

$$\frac{\pi}{2} - \delta < \angle(b_j, x, a_i), \frac{\pi}{2} - \delta < \angle(b_i, x, b_j), \text{ and } \pi - \delta < \angle(a_i, x, b_i).$$

Together, the previous two displays give

$$\angle(b_j, x, a_i) \leq \frac{\pi}{2} + \tau(\delta) \text{ and } \angle(b_i, x, b_j) \leq \frac{\pi}{2} + \tau(\delta),$$

and by a similar argument,  $\angle(a_i, x, a_j) < \frac{\pi}{2} + \tau(\delta)$ .  $\square$

**Proposition 1.5.** *Suppose  $\{M_\alpha\}_\alpha$  is a sequence of  $n$ -dimensional Alexandrov spaces with curvature  $\geq k$  that converge in the Gromov-Hausdorff topology to  $X$ . Suppose  $\{(a_i, b_i)\}_{i=1}^l$  is an  $(l, \delta, r)$ -strainer for  $y \in X$ . Let  $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^l \subset M_\alpha \times M_\alpha$  converge to  $\{(a_i, b_i)\}_{i=1}^l$ , and let  $c^\alpha \in M_\alpha$  converge to  $c \in X$ .*

*Then*

$$\left| \angle(\uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c^\alpha}) - \angle(\uparrow_y^{a_i}, \uparrow_y^c) \right| < \tau\left(\frac{1}{\alpha}|r\right) + \tau(\delta).$$

*Proof.* In general, semi-continuity of angles gives

$$\begin{aligned} \liminf_{\alpha \rightarrow \infty} \angle(\uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c^\alpha}) &\geq \angle(\uparrow_y^{a_i}, \uparrow_y^c) \text{ and} \\ \liminf_{\alpha \rightarrow \infty} \angle(\uparrow_{y^\alpha}^{b_i^\alpha}, \uparrow_{y^\alpha}^{c^\alpha}) &\geq \angle(\uparrow_y^{b_i}, \uparrow_y^c). \end{aligned} \tag{1.5.1}$$

Since  $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^l$  and  $\{(a_i, b_i)\}_{i=1}^l$  are strainers,

$$\begin{aligned} \pi - \delta &< \angle(\uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{b_i^\alpha}) \leq \angle(\uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c^\alpha}) + \angle(\uparrow_{y^\alpha}^{c^\alpha}, \uparrow_{y^\alpha}^{b_i^\alpha}) < \pi + \tau(\delta) + \tau\left(\frac{1}{\alpha}|r\right), \text{ and} \\ \pi - \delta &< \angle(\uparrow_y^{a_i}, \uparrow_y^{b_i}) \leq \angle(\uparrow_y^{a_i}, \uparrow_y^c) + \angle(\uparrow_y^c, \uparrow_y^{b_i}) < \pi + \tau(\delta), \end{aligned} \tag{1.5.2}$$

where the last upper bound on each line comes from Inequality (1.3.2) and the fact that angles are limits of comparison angles.

Combining Inequalities (1.5.1) and (1.5.2),

$$\left| \angle(\uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c^\alpha}) - \angle(\uparrow_y^{a_i}, \uparrow_y^c) \right| < \tau\left(\frac{1}{\alpha}|r\right) + \tau(\delta).$$

$\square$

If  $x$  is  $(l, \delta, r)$ -strained by  $\{(a_i, b_i)\}_{i=1}^l$ , we get an analogy with linear algebra by thinking of  $\{\uparrow_x^{a_i}\}_{i=1}^l$  as an almost orthonormal subset in  $\Sigma_x$ . This leads to

**Proposition 1.6.** *Suppose that  $x \in X$  is  $(l, \delta)$ -strained by  $\{(a_i, b_i)\}_{i=1}^l$  and  $\{(c_i, d_i)\}_{i=1}^l$ , and that  $\tilde{x} \in \tilde{X}$  is  $(l, \delta)$ -strained by  $\{(\tilde{a}_i, \tilde{b}_i)\}_{i=1}^l$  and  $\{(\tilde{c}_i, \tilde{d}_i)\}_{i=1}^l$ . In addition, suppose both sets of strainers “almost span the same subspace”, in the sense that*

$$\left| \left| \det (\cos \triangleleft (\uparrow_x^{a_i}, \uparrow_x^{c_j}))_{i,j} \right| - 1 \right| < \tau(\delta) \quad (1.6.1)$$

and

$$\left| \left| \det (\cos \triangleleft (\uparrow_{\tilde{x}}^{\tilde{a}_i}, \uparrow_{\tilde{x}}^{\tilde{c}_j}))_{i,j} \right| - 1 \right| < \tau(\delta). \quad (1.6.2)$$

Suppose further that in each space we have “almost the same change of basis matrix”, in the sense that for all  $i, j$  and for some  $\varepsilon > 0$ ,

$$\left| \triangleleft (\uparrow_x^{a_i}, \uparrow_x^{c_j}) - \triangleleft (\uparrow_{\tilde{x}}^{\tilde{a}_i}, \uparrow_{\tilde{x}}^{\tilde{c}_j}) \right| < \varepsilon. \quad (1.6.3)$$

Then given  $Y \in \Sigma_x(X)$  with

$$\left| \sum_{i=1}^l \cos \triangleleft (Y, \uparrow_x^{a_i}) - 1 \right| < \tau(\delta), \quad (1.6.4)$$

there is a  $\tilde{Y} \in \Sigma_{\tilde{x}}(\tilde{X})$  so that

$$\left| \triangleleft (Y, \uparrow_x^{a_i}) - \triangleleft (\tilde{Y}, \uparrow_{\tilde{x}}^{\tilde{a}_i}) \right| < \tau(\delta) + \tau(\varepsilon) \quad (1.6.5)$$

and

$$\left| \triangleleft (Y, \uparrow_x^{c_i}) - \triangleleft (\tilde{Y}, \uparrow_{\tilde{x}}^{\tilde{c}_i}) \right| < \tau(\delta) + \tau(\varepsilon). \quad (1.6.6)$$

*Proof.* When  $\delta = 0$ , the statement can be interpreted as a linear algebra fact. Indeed, if  $\delta = 0$ , then  $\{\uparrow_x^{a_i}\}_{i=1}^l$  and  $\{\uparrow_x^{c_j}\}_{i=1}^l$  lie in subsets  $V_a$  and  $V_c$  of  $T_x X$  that are isometric to  $\mathbb{R}^l$ , in which  $\{\uparrow_x^{a_i}\}_{i=1}^l$  and  $\{\uparrow_x^{c_j}\}_{i=1}^l$  are orthonormal bases. Inequality (1.6.1) with  $\delta = 0$ , implies that  $V_a$  and  $V_c$  are the same, since the projection  $V_a$  onto  $V_c$  carries the cube spanned by  $\{\uparrow_x^{a_i}\}_{i=1}^l$  to a parallelepiped of volume 1. Using Inequality (1.6.2), the analogous statement applies to  $\{\uparrow_{\tilde{x}}^{\tilde{a}_i}\}_{i=1}^l$  and  $\{\uparrow_{\tilde{x}}^{\tilde{c}_j}\}_{i=1}^l$ .

Inequality (1.6.4) with  $\delta = 0$  implies that  $Y$  is in the span of  $\{\uparrow_x^{a_i}\}_{i=1}^l$ . Given such a  $Y$ , there is a  $\tilde{Y}$  whose coefficients as a combination of  $\{\uparrow_{\tilde{x}}^{\tilde{a}_i}\}_{i=1}^l$  are the same as those of  $Y$  as a combination of  $\{\uparrow_x^{a_i}\}_{i=1}^l$ . That is, we get Inequality (1.6.5) when  $\delta = \varepsilon = 0$ . Inequality (1.6.3) with  $\varepsilon = 0$  implies that the change of basis matrix that carries  $\{\uparrow_x^{a_i}\}_{i=1}^l$  to  $\{\uparrow_x^{c_j}\}_{i=1}^l$  also carries  $\{\uparrow_{\tilde{x}}^{\tilde{a}_i}\}_{i=1}^l$  to  $\{\uparrow_{\tilde{x}}^{\tilde{c}_j}\}_{i=1}^l$ . Thus Inequality (1.6.6) with  $\delta = \varepsilon = 0$  follows from the  $\delta = \varepsilon = 0$  versions of Inequalities (1.6.3) and (1.6.5). By continuity, we get the result for all sufficiently small positive  $\varepsilon$  and  $\delta$ .  $\square$

**1.7. Spherical Sets and the Join Lemma.** When  $x$  is  $k$ -strained,  $\Sigma_x$  is Gromov–Hausdorff close to a space of  $\text{curv} \geq 1$  that contains a metrically embedded copy of  $\mathbb{S}^{k-1}$ . The sense in which this embedding preserves metrics is much stronger than for the isometric embeddings of Definition E. Specifically,

**Definition 1.8.** We say that an embedding  $\iota : Y \hookrightarrow X$  of a metric space  $Y$  into a metric space  $X$  is metric if and only if

$$\text{dist}_Y(y_1, y_2) = \text{dist}_X(\iota(y_1), \iota(y_2)).$$

The model space of directions for a point that is  $(m+1)$ -strained is given by the Join Lemma, which follows.

**Lemma 1.9.** (Join Lemma, [10]) Let  $X$  be an  $n$ -dimensional Alexandrov space with  $\text{curv} \geq 1$ . If  $X$  contains a metrically embedded copy of the unit  $m$ -sphere,  $\mathbb{S}^m$ , then  $E \equiv \{x \in X \mid \text{dist}(\mathbb{S}^m, x) = \frac{\pi}{2}\}$  is a metrically embedded  $(n-m-1)$ -dimensional Alexandrov space with  $\text{curv} E \geq 1$ , and  $X$  is isometric to the spherical join  $\mathbb{S}^m * E$ .

See [7] for the definition of spherical join metrics.

**Definition 1.10.** As in [1] and [30] we say an Alexandrov space  $\Sigma$  with  $\text{curv} \Sigma \geq 1$  is globally  $(m, \delta)$ -strained by pairs of subsets  $\{A_i, B_i\}_{i=1}^m$  provided

$$\begin{aligned} |\text{dist}(a_i, b_j) - \frac{\pi}{2}| &< \delta, & \text{dist}(a_i, b_i) &> \pi - \delta, \\ |\text{dist}(a_i, a_j) - \frac{\pi}{2}| &< \delta, & |\text{dist}(b_i, b_j) - \frac{\pi}{2}| &< \delta \end{aligned}$$

for all  $a_i \in A_i$  and  $b_i \in B_i$  with  $i \neq j$ .

We also consider a generalization of global strainers due to Plaut.

**Definition 1.11.** (Plaut, [23]) A set of  $2n$  points  $x_1, y_1, \dots, x_n, y_n$  in a metric space  $Y$  is called spherical if  $\text{dist}(x_i, y_i) = \pi$  for all  $i$  and  $\det[\cos \text{dist}(x_i, x_j)] > 0$ .

**Remark.** If  $x_1, \dots, x_n$  are points in  $\mathbb{S}^{n+k} \subset \mathbb{R}^{n+k+1}$ , then  $\sqrt{\det[\cos \text{dist}(x_i, x_j)]}$  is the  $n$ -dimensional volume of the parallelepiped spanned by  $\{x_1, \dots, x_n\}$ . So Plaut's condition should be viewed as a quantification of linear independence.

**Theorem 1.12.** (Plaut, [23]) If  $X$  has curvature  $\geq 1$  and contains a spherical set  $\Sigma$  of  $2(n+1)$  points, then there is a subset  $S$  of  $X$  isometric to  $\mathbb{S}^n$  such that  $\Sigma \subset S$ .

The following is a natural deformation of Plaut's condition.

**Definition 1.13.** A set of  $2n$  points  $x_1, y_1, \dots, x_n, y_n$  in a metric space  $Y$  is called  $(\delta|d)$ -almost spherical if  $\text{dist}(x_i, y_i) > \pi - \delta$  for all  $i$  and  $\det[\cos \text{dist}(x_i, x_j)] > d > 0$ .

Plaut's notion of spherical sets is related to strainers via the following result.

**Proposition 1.14.** Let  $X$  have curvature  $\geq 1$ , dimension  $n$ , and contain a  $(\delta|d)$ -almost spherical set  $S$  of  $2(m+1)$  points, for  $m < n-1$ .

There is an  $(m+1, \tau(\delta|d))$ -global strainer  $\{(a_i, b_i)\}_{i=1}^{m+1}$  for  $X$  so that

$$\text{dist}(a_i, a_j) > \frac{\pi}{2} \text{ for } i \neq j.$$

Moreover, for all  $\kappa \in (0, \frac{\pi}{4})$ , if  $\delta$  is sufficiently small compared to  $d$  and  $\kappa$ , there is a nonempty set  $E \subset X$  so that for all  $e \in E$

$$\frac{\pi}{2} < \text{dist}(e, a_i) < \frac{\pi}{2} + \kappa,$$

and

$$\left| \text{dist}(e, b_i) - \frac{\pi}{2} \right| < \kappa.$$

*Proof.* First we consider the rigid case when  $X$  contains an isometric copy of  $\mathbb{S}^m$ . Perturbing an orthonormal basis, one sees that  $X$  contains a global  $(m+1, \delta)$ -strainer  $\{(a_i, b_i)\}_{i=1}^{m+1} \subset \mathbb{S}^m$  so that

$$\text{dist}(a_i, a_j) > \frac{\pi}{2} \text{ for } i \neq j.$$

We can also find a point  $h \in \mathbb{S}^m$  with

$$\text{dist}(a_i, h) > \frac{\pi}{2} \text{ for all } i.$$

By the Join Lemma,  $\tilde{E} \equiv \{x \in X \mid \text{dist}(\mathbb{S}^m, x) = \frac{\pi}{2}\}$  is a metrically embedded  $(n-m-1)$ -dimensional Alexandrov space with  $\text{curv} \tilde{E} \geq 1$ , and  $X$  is isometric to the join  $\mathbb{S}^m * \tilde{E}$ .

Combining this with  $\text{dist}(a_i, h) > \frac{\pi}{2}$ , it follows that for all  $\tilde{e} \in \tilde{E}$ , the interior of the segment  $\tilde{e}h$  is further than  $\frac{\pi}{2}$  from all the points  $a_i$ . For any fixed  $\kappa \in (0, \frac{\pi}{4})$ , we set

$$E = \left\{ \tilde{e}h \left( \frac{\kappa}{2} \right) \mid \tilde{e} \in \tilde{E} \right\}.$$

This completes the proof in the rigid case. The general case follows from the rigid case, Theorem 1.12, Lemma 1.9, and a proof by contradiction.  $\square$

**1.15. Gromov Packing.** We make use a version of Gromov's Packing Lemma. Its closest relative in the literature, as far as we know, is on page 230 of [31].

**Lemma 1.16.** (*Gromov's Packing Lemma*) *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq k$  for some  $k \in \mathbb{R}$ . There are positive constants  $\mathfrak{o}(n, k)$  and  $r_0(n, k)$  with the following property. For all  $r \in (0, r_0)$ , any compact subset of  $A \subset X$  contains a finite subset  $\{a_i\}_{i \in I}$  so that*

- $A \subset \cup_i B(a_i, r)$ , and
- the order of the cover  $\{B(a_i, 3r)\}_i$  is  $\leq \mathfrak{o}$ .

In the Riemannian case, this follows from relative volume comparison, so one only needs the corresponding lower bound on Ricci curvature. Since relative volume comparison holds for rough volume in Alexandrov spaces, the proof in [31] yields, with minor modifications, Lemma 1.16.

## 2. RIEMANNIAN SUBMANIFOLDS OF ALEXANDROV SPACES

Here we establish several results that are relevant to smooth, isometric embeddings of Riemannian manifolds into Alexandrov spaces. In the first subsection, we show that the unit tangent sphere of each point  $p \in S$  metrically embeds into the space of directions of  $p$  in  $X$ . In the second subsection, we prove Theorem 2.14, which gives local Alexandrov models of the vector bundles of the TNST.

### 2.1. Riemannian versus Alexandrov Spaces of Directions.

**Definition 2.2.** ([1], page 48) Let  $c : [-a, a] \rightarrow \mathbb{R}$  be a unit speed curve in an Alexandrov space  $X$ . The right and left derivatives of  $c$  at 0 are

$$c'_+(0) \equiv \lim_{t \rightarrow 0^+} \uparrow_{c(0)}^{c(t)} \quad \text{and} \quad c'_-(0) \equiv \lim_{t \rightarrow 0^-} \uparrow_{c(0)}^{c(t)},$$

provided the limits exist and are single directions.

**Proposition 2.3.** Let  $f : [0, b] \rightarrow \mathbb{R}$  be continuous and  $C^1$  on  $(0, b]$ . If  $f'$  has a continuous extension to  $[0, b]$ , then  $f$  is also differentiable at 0.

*Proof.* For  $x \in (0, b)$  we have a number  $c \in (0, x)$  so that

$$\frac{f(x) - f(0)}{x} = f'(c).$$

Since  $f'$  has a continuous extension to 0, it follows that  $f'(0)$  exists.  $\square$

**Proposition 2.4.** Let  $S$  be a Riemannian manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ . For every  $\varepsilon > 0$  there is a  $\delta > 0$  so that for distinct  $a, b \in S$  with  $\text{dist}^S(a, b) < \delta$ ,

$$\left| \frac{\text{dist}^S(a, b)}{\text{dist}^X(a, b)} - 1 \right| < \varepsilon.$$

*Proof.* Let  $\gamma$  be a unit speed geodesic from  $a$  to  $b$  in  $S$ . Let  $\gamma_{\text{mid}}^a$  and  $\gamma_{\text{mid}}^b$  be the unit speed reparameterizations of  $\gamma$  that go from the midpoint,  $m_p$ , of  $\gamma$  to  $a$  and  $b$ , respectively. Then

$$(\gamma_{\text{mid}}^a, \gamma_{\text{mid}}^b) : \left[ 0, \frac{\text{dist}^S(a, b)}{2} \right] \rightarrow S \times S$$

is a speed  $\sqrt{2}$  geodesic in  $S \times S$  from  $(m_p, m_p)$  to  $(a, b)$  with respect to the product metric.

Since the derivative of  $\text{dist}^S \circ (\gamma_{\text{mid}}^a, \gamma_{\text{mid}}^b)$  is constant and equal to 2, it follows from Definition E and Proposition 2.3 that the derivative of  $\text{dist}^X \circ (\gamma_{\text{mid}}^a, \gamma_{\text{mid}}^b)$  is 2 at 0. Combining this with Definition E, it follows that if  $\text{dist}^S(a, b)$  is sufficiently small, then

$$\text{dist}^S(a, b) \geq \text{dist}^X(a, b) \geq \text{dist}^S(a, b) - o(\text{dist}^S(a, b)) \quad (2.4.1)$$

and

$$1 \leq \frac{\text{dist}^S(a, b)}{\text{dist}^X(a, b)} \leq \frac{\text{dist}^S(a, b)}{\text{dist}^S(a, b) - o(\text{dist}^S(a, b))}.$$

The result follows.  $\square$

**Proposition 2.5.** Let  $S$  be a Riemannian manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$  with curvature  $\geq -1$ . Along an intrinsic unit speed geodesic  $c$  of  $S$ ,  $c'_+(0)$  exists.

*Proof.* If not, then we have  $v_{t_i} \in \left( \uparrow_{c(0)}^{c(t_i)} \right)_X$ ,  $w_{s_i} \in \left( \uparrow_{c(0)}^{c(s_i)} \right)_X$ ,  $v = \lim_{t_i \rightarrow 0^+} v_{t_i}$ ,  $w = \lim_{s_i \rightarrow 0^+} w_{s_i}$ , and  $v \neq w$ .

By Proposition 2.4, for  $\varepsilon$ ,  $s_i$ , and  $t_i$  sufficiently close to 0,

$$\text{dist}^X(c(-\varepsilon), c(s_i)) = \varepsilon + s_i + o(\varepsilon + s_i), \text{ and} \quad (2.5.1)$$

$$\text{dist}^X(c(-\varepsilon), c(t_i)) = \varepsilon + t_i + o(\varepsilon + t_i). \quad (2.5.2)$$

We let  $\gamma_{v_{t_i}}$  and  $\gamma_{w_{s_i}}$  be the  $X$ -geodesics so that  $\gamma'_{v_{t_i}}(0) = v_{t_i}$  and  $\gamma'_{w_{s_i}}(0) = w_{t_i}$ . Since  $v \neq w$ , there is a  $\beta \in (0, \frac{\pi}{2})$  so that for all but finitely many  $i$ , either

$$\angle(\gamma_{v_{t_i}}, \left(\uparrow_{c(0)}^{c(-\varepsilon)}\right)_X) < \pi - \beta \text{ or } \angle(\gamma_{w_{s_i}}, \left(\uparrow_{c(0)}^{c(-\varepsilon)}\right)_X) < \pi - \beta.$$

The argument is the same in both cases, so suppose the former holds. Hinge comparison gives

$$\begin{aligned} \text{dist}^X(c(-\varepsilon), c(t_i)) &\leq \varepsilon - t_i \cos(\pi - \beta) + \tau(t_i|\varepsilon) \\ &= \varepsilon + t_i \cos(\beta) + \tau(t_i|\varepsilon) \end{aligned}$$

Since both the previous inequality and (2.5.2) hold for all sufficiently small  $\varepsilon, t_i > 0$ , we have a contradiction.  $\square$

**Proposition 2.6.** *Let  $S$  be a Riemannian manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ , and let  $c$  be an intrinsic unit speed geodesic of  $S$ . If  $a$  and  $c(0)$  are sufficiently close distinct points of  $S$ , then  $\text{dist}_a^X \circ c$  is differentiable at 0, and*

$$\begin{aligned} (\text{dist}_a^X \circ c)'(0) &= D_{c'_+(0)}(\text{dist}_a^X) \\ &= -\cos\left(\angle(c'_+(0), \left(\uparrow_{c(0)}^a\right)_X)\right). \end{aligned}$$

*Proof.* Since  $S \hookrightarrow X$  is smooth and isometric,  $\text{dist}_a^X$  is  $C^1$  at  $c(0)$ . Since  $\text{dist}_a^X$  is also a distance function of  $X$ , it is directionally differentiable, and

$$D_{c'_+(0)}(\text{dist}_a^X) = -\cos\left(\angle((c)'_+(0), \left(\uparrow_{c(0)}^a\right)_X)\right).$$

On the other hand, writing  $d_a^X$  for  $\text{dist}_a^X$ ,

$$\begin{aligned} &\left| (d_a^X \circ c)'(0) - \lim_{t \rightarrow 0^+} D_{\left(\uparrow_{c(0)}^{c(t)}\right)_X} (d_a^X) \right|^2 \\ &= \left| \lim_{s \rightarrow 0^+} \frac{1}{s} \{d_a^X(c(s)) - d_a^X(c(0))\} - \lim_{t \rightarrow 0^+} \lim_{s \rightarrow 0^+} \frac{1}{s} \left\{ d_a^X\left(\gamma_{\left(\uparrow_{c(0)}^{c(t)}\right)_X}(s)\right) - d_a^X(c(0)) \right\} \right|^2 \\ &= \lim_{t \rightarrow 0^+} \left| \lim_{s \rightarrow 0^+} \frac{1}{s} \left\{ d_a^X(c(s)) - d_a^X\left(\gamma_{\left(\uparrow_{c(0)}^{c(t)}\right)_X}(s)\right) \right\} \right|^2 \\ &\leq \lim_{t \rightarrow 0^+} \left| \lim_{s \rightarrow 0^+} \frac{1}{s} \left\{ \text{dist}^X\left(c(s), \gamma_{\left(\uparrow_{c(0)}^{c(t)}\right)_X}(s)\right) \right\} \right|^2 \\ &\leq \lim_{t \rightarrow 0^+} \lim_{s \rightarrow 0^+} \frac{1}{s^2} \left( 2s^2 - 2s^2 \cos \angle\left(\left(\uparrow_{c(0)}^{c(s)}\right)_X, \left(\uparrow_{c(0)}^{c(t)}\right)_X\right) + O(s^4) \right), \text{ by Proposition 2.4} \\ &= \lim_{t \rightarrow 0^+} \lim_{s \rightarrow 0^+} \left( 2 - 2 \cos \angle\left(\left(\uparrow_{c(0)}^{c(s)}\right)_X, \left(\uparrow_{c(0)}^{c(t)}\right)_X\right) \right) \\ &= 0, \text{ since } c'_+(0) \text{ exists.} \end{aligned}$$

Thus

$$(\text{dist}_a^X \circ c)'(0) = \lim_{t \rightarrow 0^+} D_{\left(\uparrow_{c(0)}^{c(t)}\right)_X} (\text{dist}_a^X) = D_{c'_+(0)} (\text{dist}_a^X),$$

as claimed.  $\square$

**Proposition 2.7.** *Let  $S$  be a Riemannian manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ . For  $p \in S$ , let  $T_p^1 S$  be the Riemannian unit tangent sphere to  $S$  at  $p$ , and for  $v \in T_p^1 S$ , let  $c_v(t) = \exp_p^S(tv)$ . Then the map*

$$\begin{aligned} \iota &: T_p^1 S \longrightarrow \Sigma_p X \\ \iota &: v \mapsto (c_v)'_+(0)|_{t=0} \end{aligned}$$

is a metric embedding.

In particular, for a geodesic  $c$  of  $S$ ,

$$\angle_X(c'_+(0), c'_-(0)) = \pi.$$

*Proof.* Let  $v, w \in T_p^1 S$ , and let  $a$  be a point on  $c_w$  that is different from  $p$ . Then

$$\begin{aligned} -\cos(\angle_S(v, w)) &= D_v \text{dist}_a^S(\cdot) \\ &= D_v \text{dist}_a^X(\cdot) + \tau(\text{dist}_a^S(c_v(0))), \text{ by Definition E} \\ &= (\text{dist}_a^X \circ c_v)'(0) + \tau(\text{dist}_a^S(c_v(0))) \\ &= -\cos\left(\angle\left((c_v)'_+(0), \left(\uparrow_{c_v(0)}^a\right)_X\right)\right) + \tau(\text{dist}_a^S(c_v(0))), \text{ by Proposition 2.6.} \end{aligned}$$

Taking the limit as  $a \rightarrow c_v(0) = c_w(0)$  and appealing to Proposition 2.5, we conclude

$$-\cos(\angle_S(v, w)) = -\cos(\angle_X((c_v)'_+(0), (c_w)'_+(0))),$$

as desired.  $\square$

The metric embedding  $\iota : T_p^1 S \longrightarrow \Sigma_p X$  induces a metric embedding  $T_p S \longrightarrow T_p X$ . From here on, we will make no notational distinction between  $T_p^1 S$  and  $T_p S$  and their images under these embeddings.

So for example we set

$$\iota(T_p^1 S) = \Sigma_p S \subset \Sigma_p X,$$

and for  $c_v(t) = \exp_p^S(tv)$ , we have

$$c'_v(0) = \lim_{t \rightarrow 0^+} \left(\uparrow_{c(0)}^{c(t)}\right)_X, \quad (2.7.1)$$

where all vectors are directions in  $\Sigma_p X$ .

**2.8. How to cover  $S \hookrightarrow X$ .** In the main result of this subsection, Theorem 2.14, we construct a cover  $\mathcal{O}$  of  $X$  that decomposes,  $\mathcal{O} = \bigcup_{S \in \mathcal{S}^{\text{ext}}} \mathcal{O}^S$ , into subcollections  $\mathcal{O}^S$ —one for

each element of  $\mathcal{S}^{\text{ext}}$ . The elements of  $\mathcal{O}^X$  are  $(n, \delta)$ -strained and are contained in the top stratum. A posteriori, their union is a Gromov-Hausdorff approximation of the sets  $G_\gamma$  of Part 3 of the TNST. Similarly, the elements of each  $\mathcal{O}^{S_i}$  are  $(\dim S_i, \tilde{\delta})$ -strained by points of  $S_i$ , and their union will be Gromov-Hausdorff close to the sets  $\mathcal{U}_\gamma^{S_i}$  of Part 2 of the

TNST. In fact, the strainers for these sets will also give us local Alexandrov versions of the diffeomorphism of Part 4 and the submersions of Part 2 of the TNST.

The statement of Theorem 2.14 is rather technical, so we first prove a series of preliminary results beginning with the following application of Equation (2.7.1).

**Lemma 2.9.** *Let  $(S, g)$  be a Riemannian  $k$ -manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ , and let  $K$  be a compact subset of  $S$ . Given  $\varepsilon, \tilde{\delta} > 0$  there is an  $r_0 > 0$  so that for all  $r \in (0, r_0)$  there is a  $\rho > 0$  with the following properties.*

1. *For all  $p \in K$ ,  $B(p, 3\rho)$  is  $(k, \tilde{\delta}, r)$ -strained in  $X$  by points  $\{(a_i, b_i)\}_{i=1}^k$  contained in  $S \times S$ .*
2. *For all  $i$ , and for all  $x \in B(p, 3\rho) \cap S$ ,*

$$\angle((\uparrow_x^{a_i})_X, T_x^1 S) < \varepsilon.$$

*Proof.* First we prove the existence of  $r_0$  for a single point  $p \in S$ . Take  $\{v_i\}_{i=1}^k \subset T_p S$  to be an orthonormal basis. Let  $c_{v_i}$  be the geodesic in  $S$  with  $(c_{v_i})'(0) = v_i$ . Choose  $r \in (0, \frac{1}{4}\text{inj}_p(S))$ , and set  $a_i = c_{v_i}(4r)$  and  $b_i = c_{v_i}(-4r)$ .

Since  $\{v_i\}_{i=1}^k$  is an orthonormal basis for  $T_p S$ , it follows from Proposition 2.7 that  $\{v_i\}_{i=1}^k$  is an orthonormal subset of  $\Sigma_p X$ . So if  $r$  is sufficiently small, then

$$\{(a_i, b_i)\}_{i=1}^k \text{ is a } (k, \tilde{\delta}, r)\text{-strainer for a neighborhood } N_X \text{ of } p \text{ in } X, \quad (2.9.1)$$

giving us Property 1 at  $p$ .

By Equation (2.7.1), given  $\eta \in (0, \varepsilon)$ ,

$$\angle((\uparrow_p^{a_i})_X, v_i) < \eta, \quad (2.9.2)$$

if  $r$  is small enough.

Inequality (2.9.2) implies

$$-1 \leq D_{v_i} \text{dist}_{a_i}^X(\cdot) = -\cos\left(\angle((\uparrow_p^{a_i})_X, v_i)\right) \leq -1 + \tau(\eta). \quad (2.9.3)$$

Set

$$V_i(x) = (\uparrow_x^{a_i})_S.$$

Part 1 of the definition of a smooth, isometric embedding gives that  $D_{V_i(x)} \text{dist}_{a_i}^X(\cdot)$  is close to  $D_{v_i} \text{dist}_{a_i}^X(\cdot)$  if  $x$  is close to  $p$ . Combining this with Inequality (2.9.3) gives

$$D_{V_i(x)} \text{dist}_{a_i}^X(\cdot) = -1 + \tau(\eta)$$

for all  $x$  in a neighborhood of  $p$ . A direction  $w \in \Sigma_x X$  for which  $D_w \text{dist}_{a_i}^X(\cdot) = -1 + \tau(\eta)$  must be within  $\tau(\eta)$  of  $(\uparrow_x^{a_i})_X$ . So viewing  $V_i(x) \in \Sigma_x X$ , it follows that

$$\angle((\uparrow_x^{a_i})_X, V_i(x)) < \tau(\eta).$$

Since we also have  $V_i(x) \in T_x S$ ,

$$\angle((\uparrow_x^{a_i})_X, T_x^1 S) < \tau(\eta),$$

giving us Property 2 at  $p$ .



The existence of an  $r_0$  that works uniformly throughout a compact subset  $K$  of  $S$  follows from the stability of Properties 1 and 2. Indeed, if  $\{p_i\}_{i=1}^\infty \subset K$  converges to  $p_\infty \in K$ , then we have shown that Properties 1 and 2 hold for  $p_\infty$ . It follows that they also hold for all but finitely many of the  $\{p_i\}_i$ s with the corresponding constants divided by 2. The existence of a uniform  $r_0$  follows from this and a contradiction argument.  $\square$

Applying Lemmas 1.16 and 2.9 to a precompact open subset of  $S$ , we get the following corollary.

**Corollary 2.10.** *Let  $(S, g)$  be a Riemannian  $k$ -manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ . Let  $O \subset S$  be a precompact open subset of  $S$ . There is an  $\mathfrak{o} > 0$  so that given  $\varepsilon, \tilde{\delta} > 0$  there is an  $r > 0$ , a  $\rho_0 \in (0, r)$ , and, for all  $\rho \in (0, \rho_0)$ , a finite open cover  $\mathcal{O} \equiv \{B_j(\rho)\}_j$  of  $O$  by  $\rho$ -balls of  $X$  for which the corresponding  $3\rho$ -balls have the following properties.*

1. *Each  $B_j(3\rho)$  is  $(k, \tilde{\delta}, r)$ -strained in  $X$  by  $\{(a_i^j, b_i^j)\}_{i=1}^k$  with  $a_i^j, b_i^j \in S$ .*
2. *For all  $i, j$  and for all  $x \in B_j(3\rho) \cap S$ ,*

$$\angle \left( \left( \uparrow_x^{a_i^j} \right)_X, T_x S \right) < \varepsilon. \quad (2.10.1)$$

3. *The order of  $\{B_j(3\rho)\}_j$  is  $\leq \mathfrak{o}$ .*

**Lemma 2.11.** *Let  $(S, g)$  be a Riemannian  $k$ -manifold that is smoothly and isometrically embedded in an Alexandrov space  $X$ . Given any  $p \in S$  and  $\varepsilon, \tilde{\delta} > 0$ , let  $\{(a_i, b_i)\}_{i=1}^k$  be as in the previous lemma. There is an  $\eta_0 > 0$  so that  $\text{dist}^X(S, \cdot)$  is  $(1 - \varepsilon)$ -regular on  $B(p, 2\eta_0) \setminus S$ .*

*In fact, for all  $x \in B(p, 2\eta_0) \setminus S$ , there is a  $V^S \in \Sigma_x$  so that*

$$D_{V^S} \text{dist}^X(S, \cdot) > 1 - \varepsilon, \quad (2.11.1)$$

and

$$D_{V^S} \text{dist}_{a_i}^X \leq \tau(\tilde{\delta}) + \tau(\rho|r). \quad (2.11.2)$$

*Proof.* By Proposition 2.7, for all  $p \in S$ , we have that  $\Sigma_p S$  is a metric copy of  $\mathbb{S}^{\dim(S)-1} \subset \Sigma_p X$ . By the Join Lemma 1.9,  $\Sigma_p X$  is isometric to  $\mathbb{S}^{\dim(S)-1} * E$ , where  $E$  is a compact Alexandrov space of curvature  $\geq 1$ . It follows that  $T_p X$  splits orthogonally as

$$T_p X = T_p S \oplus C(E).$$

Under the convergence  $\lim_{\lambda \rightarrow \infty} (\lambda X, p) = (T_p X, *)$ , we have  $\lim_{\lambda \rightarrow \infty} (\lambda S, p) = (T_p S, *)$ . The result holds with  $X, S$ , and  $\{(a_i, b_i)\}_{i=1}^k$  replaced by  $T_p X, T_p S$ , and  $\{(\uparrow_p^{a_i}, \uparrow_p^{b_i})\}_{i=1}^k$ . The stability of regular points gives us that for all  $x \in B(p, 2\eta_0) \setminus S$ , there is a  $V^S \in \Sigma_x$  so that

$$D_{V^S} \text{dist}^X(S, \cdot) > 1 - \varepsilon.$$

Since  $\uparrow_p^{a_i} = \lim_{\lambda \rightarrow \infty} p a_i \left( \frac{1}{\lambda} \right)$  and  $\uparrow_p^{b_i} = \lim_{\lambda \rightarrow \infty} p b_i \left( \frac{1}{\lambda} \right)$ , it follows that (2.11.2) holds with  $\{(a_i, b_i)\}_{i=1}^k$  replaced with  $\{(\tilde{a}_i, \tilde{b}_i)\}_{i=1}^k$ , where  $\tilde{a}_i \equiv p a_i \left( \frac{1}{\lambda} \right)$  and  $\tilde{b}_i \equiv p b_i \left( \frac{1}{\lambda} \right)$ . Since the directional derivatives of  $\text{dist}_{a_i}$  and  $\text{dist}_{\tilde{a}_i}$  are nearly the same at  $p$ , (2.11.2) also holds.  $\square$

**Lemma 2.12.** *Let  $N$  be an element of  $\mathcal{S}$ , and let  $S \in \mathcal{S}$  be contained in  $\bar{N}$  and not equal to  $N$ . Given  $p \in S$  and  $\varepsilon, \tilde{\delta} > 0$ , let  $\{(a_i, b_i)\}_{i=1}^{\dim(S)}$  be as in Lemma 2.9. If  $\nu$  is sufficiently small, then for all  $\tilde{p} \in B(p, 2\nu) \cap N$ , the following hold.*

1.  $\tilde{p}$  is  $(\dim(N), \tau(\tilde{\delta}, \nu))$ -strained in  $X$  by  $\{(a_i, b_i)\}_{i=1}^{\dim(S)}$  and  $(\dim(N) - \dim(S))$  pairs of points of  $N$ ,  $\left\{ \left( a_j^{\tilde{p}}, b_j^{\tilde{p}} \right) \right\}_{j=\dim(S)+1}^{\dim(N)}$ .
2. At every  $x \in N$  that is close enough to  $\tilde{p}$ ,

$$\triangleleft \left( \left( \uparrow_x^{a_i} \right)_X, T_x N \right) < \varepsilon, \quad (2.12.1)$$

and

$$\triangleleft \left( \left( \uparrow_x^{a_j^{\tilde{p}}} \right)_X, T_x N \right) < \varepsilon. \quad (2.12.2)$$

3. For  $V^S$  as in Lemma 2.11,

$$\triangleleft \left( \left( \uparrow_x^{a_{\dim(S)+1}^{\tilde{p}}} \right)_X, V^S \right) < \tau(\tilde{\delta}, \varepsilon) + \tau(\rho|r), \quad (2.12.3)$$

where  $x \in B(\tilde{p}, \rho)$ , and  $B(\tilde{p}, \rho)$  is  $(\dim(N), \tau(\tilde{\delta}, \nu), r)$ -strained in  $X$  by

$$\left\{ \{(a_i, b_i)\}_{i=1}^{\dim(S)}, \left\{ \left( a_j^{\tilde{p}}, b_j^{\tilde{p}} \right) \right\}_{j=\dim(S)+1}^{\dim(N)} \right\}.$$

4. If  $N$  is the top stratum, that is, if  $N = X \setminus \cup_{S \in \mathcal{S}} S$ , then these same assertions hold except that in Inequality (2.12.3) we replace  $\tau(\tilde{\delta}, \varepsilon)$  with  $\tau(\delta)$ , and in Part 1,  $\tilde{p}$  is only  $(\dim(N), \delta)$ -strained.

**Remark on all things  $\delta$ .** *The distinction between  $\tau(\tilde{\delta}, \varepsilon)$ ,  $\tau(\delta)$  and  $\delta$  in Part 4 is not merely academic. In fact,  $\tilde{\delta}, \varepsilon$  and  $\rho$  can be arbitrarily small in Corollary 2.10, whereas the  $\delta$  such that all points of our top stratum are  $(n, \delta)$ -strained is determined by  $X$ , and is therefore fixed.*

*Proof.* Since strainers are stable, every point  $\tilde{p} \in B(p, \nu) \cap N$  is  $(\dim(S), \tau(\tilde{\delta}, \nu))$ -strained in  $X$  by  $\{(a_i, b_i)\}_{i=1}^{\dim(S)}$ . Combining this with Lemma 2.9 and the fact that every point of  $N$  is  $(\dim(N), 0)$ -strained and not  $(\dim(N) + 1, \delta)$ -strained gives us Inequality (2.12.1), if we choose  $\max\{\tilde{\delta}, \nu, \varepsilon\} \ll \delta$ .

The existence of  $\left\{ \left( a_j^{\tilde{p}}, b_j^{\tilde{p}} \right) \right\}_{j=1}^{\dim(N) - \dim(S)}$  follows from the fact that every point of  $N$  is  $(\dim(N), 0)$ -strained, and the proof of Lemma 2.9 gives us Inequality (2.12.2).

It follows from Inequalities (2.11.1) and (2.11.2) that  $\left( a_{\dim(S)+1}^{\tilde{p}}, b_{\dim(S)+1}^{\tilde{p}} \right)$  can be chosen so that  $\triangleleft \left( \left( \uparrow_x^{a_{\dim(S)+1}^{\tilde{p}}} \right)_X, V^S \right) < \tau(\tilde{\delta}, \varepsilon) + \tau(\rho|r)$ , provided  $\nu \in (0, \eta_0)$ , where  $\eta_0$  is as in Lemma 2.11.  $\square$

For  $a \in X$  and  $\eta > 0$ , we define  $g_a : X \rightarrow \mathbb{R}$  by

$$g_a(y) = \frac{1}{\text{vol}(B(a, \eta))} \int_{z \in B(a, \eta)} \text{dist}(y, z). \quad (2.12.4)$$

Differentiation under the integral and the directional differentiability of distance functions gives the following.

**Proposition 2.13.** *If  $X$  is a Riemannian manifold, then  $g_a$  is  $C^1$ , and, in general, for any  $v \in T_y X$ ,*

$$D_v(g_a) = \frac{1}{\text{vol}(B(a, \eta))} \int_{z \in B(a, \eta)} D_v(\text{dist}(\cdot, z)).$$

If  $B(x, \rho)$  is  $(l, \delta, r)$ -strained by  $\{(a_i, b_i)\}_{i=1}^l$ , let  $p : B(x, \sigma) \rightarrow \mathbb{R}^l$  be defined by

$$p(y) \equiv (g_{a_1}(y), \dots, g_{a_l}(y)). \quad (2.13.1)$$

It is of course true that  $p$  depends on  $\eta$ ; however, we adopt the convention that all assertions about the maps  $p$  defined in (2.13.1) have the added implicit assumption that  $\eta$  is sufficiently small.

Let  $X$  and  $\mathcal{S}$  be as in Theorem B. Recall that

$$\mathcal{S}^{\text{ext}} \equiv \mathcal{S} \cup (X \setminus \cup_{S \in \mathcal{S}} S).$$

For an element  $S \in \mathcal{S}^{\text{ext}}$ , we write  $\bar{S}$  for the closure of  $S$  and set

$$Bd(S) \equiv \bar{S} \setminus S.$$

Note that  $\dim(Bd(S))$  can be  $\leq \dim(S) - 2$ ; in particular,  $\bar{S}$  need not be a manifold with boundary.

**Theorem 2.14.** *Let  $X, \mathcal{K}$ , and  $\mathcal{N}$  satisfy the hypotheses of Theorem B. Given  $\varepsilon, \tilde{\delta} > 0$ , there are  $\rho_0^X, \rho_0^{S_i} > 0$ , and, for all  $\rho^X \in (0, \rho_0^X)$  and  $\rho^{S_i} \in (0, \rho_0^{S_i})$ , there are collections of open sets  $\mathcal{O}^X \equiv \{B_k^X(\rho^X)\}_k$  and  $\{\mathcal{O}^{S_i}\}_i \equiv \left\{ \{B_j^{S_i}(\rho^{S_i})\}_{j \in I_{S_i}} \right\}_i$  where each  $B_k^X(\rho^X)$  is a metric  $\rho^X$ -ball of  $X$  and each  $B_j^{S_i}(\rho^{S_i})$  is a metric  $\rho^{S_i}$ -balls of  $X$  with the following properties.*

1. *Set  $O_i \equiv \cup_{j \in I_{S_i}} B_j^{S_i}(\rho^{S_i}) \cap S_i$ . Corollary 2.10 holds for each  $O_i$ .*
2. *There is an  $r > 0$  so that each  $B_k^X(3\rho^X)$  is  $(n, \delta, r)$ -strained.*
3. *If  $S_i \in \mathcal{K}$ , then  $\mathcal{O}^{S_i}$  is a cover of  $S_i$ .*
4. *For  $N \in \mathcal{S}^{\text{ext}}$ , if  $Bd(N) = \amalg_{n_i} S_{n_i}$ , then  $\mathcal{O}^N$  together with the  $\mathcal{O}^{S_{n_i}}$ s is a cover of  $N$ .*
5. *Let  $S$  and  $N$  be elements of  $\mathcal{S}^{\text{ext}}$ , and let  $S$  be a subset of  $Bd(N)$ . For all  $x \in (\cup \mathcal{O}^N) \cap (\cup \mathcal{O}^S \setminus B(S, \frac{\rho^S}{2}))$ , there is a  $B_k^N(\rho^N) \in \mathcal{O}^N$  and a  $B_{j(k)}^S(\rho^S) \in \mathcal{O}^S$  so that*

$$\begin{aligned} x &\in B_k^N(\rho^N), \\ B_k^N(3\rho^N) &\subseteq B_{j(k)}^S(\rho^S), \text{ and} \\ \pi_{\dim(S)} \circ p_k^N &= p_{j(k)}^S, \end{aligned} \quad (2.14.1)$$

where  $p_k^N : B_k(\rho^N) \rightarrow \mathbb{R}^{\dim(N)}$  and  $p_{j(k)}^S : B_{j(k)}^S(\rho^S) \rightarrow \mathbb{R}^{\dim(S)}$  are defined as in (2.13.1), and  $\pi_{\dim(S)} : \mathbb{R}^{\dim(N)} \rightarrow \mathbb{R}^{\dim(S)}$  is projection onto the first  $\dim(S)$  factors.

6. For  $N \in \mathcal{S}^{\text{ext}}$  and  $S \in \text{Bd}(N)$ , let  $B_k^N(3\rho^N) \subset B_{j(k)}^S(\rho^S)$  be as in Part 5. We can choose the strainers of  $B_k(3\rho^N)$  so that Part 3 of Lemma 2.12 holds.

**Remark 2.15.** The statement of Part 5 of Theorem 2.14 is rather technical, but has the virtue of giving local Alexandrov versions of Parts 4 and 5 of the TNST.

Next we define the Generation Number of each  $S \in \mathcal{S}$ . It is dual to the concept of Ancestor Number that appears on page 5. Recall that we partially ordered the  $S \in \mathcal{S}^{\text{ext}}$  by declaring that  $S_1 < S_2$  if  $S_1 \subsetneq \bar{S}_2$ , where  $\bar{S}_2$  is the closure of  $S_2$ . We call the number,  $a$ , the Generation Number of  $S \in \mathcal{S}^{\text{ext}}$  if  $a$  is the length of the largest chain

$$S_0 < S_1 < \cdots < S_a$$

with  $S = S_a$  and  $S_0 = \bar{S}_0$ . Let  $\mathcal{S}_j$  be the collection of all  $S \in \mathcal{S}^{\text{ext}}$  that have generation number  $j$ .

*Proof of Theorem 2.14.* The proof is by induction on the Generation Number. If  $S \in \mathcal{S}$  has generation number 0, then we get the desired cover  $\mathcal{O}^S$  from Corollary 2.10.

Suppose by induction that we have constructed the desired cover  $\mathcal{O}(k)$  of the union of the elements of  $\cup_{j=0}^k \mathcal{S}_j$ , and  $3\mathcal{O}(k)$  is the corresponding cover by balls on three times the radii. For  $N \in \mathcal{S}_{k+1}$ , let

$$J_N \equiv \{j \in I \mid S_j \subset \text{Bd}(N) \text{ and } S_j \in \mathcal{S}\}.$$

We apply Lemma 2.12 to obtain a cover  $\mathcal{O}^{N, \text{pre}}$  of

$$\left\{ (\cup 3\mathcal{O}(k)) \setminus \cup_{j \in J_N} B\left(S_j, \frac{\rho^{S_j}}{2}\right) \right\} \cap N$$

that satisfies Parts 5 and 6. Since  $\left\{ (\cup 3\mathcal{O}(k)) \setminus \cup_{j \in J_N} B\left(S_j, \frac{\rho^{S_j}}{2}\right) \right\} \cap N$  is precompact in  $N$ , we can take  $\mathcal{O}^{N, \text{pre}}$  to be a finite cover. We then apply Corollary 2.10 with  $O = N \setminus \cup_{j \in J_N} B\left(S_j, \frac{\rho^{S_j}}{2}\right)$  to get the desired cover of  $N$ . Since there are only finitely many  $N \in \mathcal{S}_{k+1}$ , this completes the induction step.  $\square$

### 3. LOCAL STRAIN AND CONVEX STRUCTURE OF ALEXANDROV SPACES

The main result of this section is Theorem 3.4. It provides local versions of the vector bundles of Part 2 of the TNST over each member of the open cover of Theorem 2.14. In the next section we show that the projections of our local vector bundles are  $C^1$ -close on their intersections, and in Section 5 we state a theorem about gluing together  $C^1$ -close submersions.

Theorem 3.4 is proven by combining Theorem 2.14 with Perelman's remarkable concavity construction. We start with a review of Perelman Concavity.

**Proposition 3.1.** (Perelman Concavity, [21]) *Let  $X$  be an  $n$ -dimensional Alexandrov space of curvature  $\geq -1$ . Suppose  $q, p \in X$  satisfy  $\text{dist}(q, p) = d$ , and for some  $\eta, v > 0$ ,  $\text{vol}(B(p, \eta)) \geq v$ . Then there is a  $\delta > 0$  and a smooth increasing function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  so that*

$$f_p(x) = \frac{1}{\text{vol}(B(p, \eta))} \int_{z \in B(p, \eta)} \psi \circ \text{dist}(x, z)$$

is strictly  $-1$ -concave on  $B(q, \delta)$ .

Moreover, if  $\psi$  satisfies  $\frac{1}{2} < \psi' \leq 2$ , then  $f_p$  is directionally differentiable and satisfies

$$|D_V f_p| \leq 2 \quad (3.1.1)$$

for all directions  $v$ .

*Proof.* The idea is to choose  $\psi$  to have a very negative second derivative and so that  $\frac{1}{2} < \psi' \leq 2$  on a very small interval around the number  $\text{dist}(p, q)$ .

Indeed, the lower curvature bound gives us a  $\lambda > 0$  so that for any  $z \in B(p, \eta)$ ,  $x$  near  $q$ , and a direction  $w \in \Sigma_x$ ,

$$\psi \circ \text{dist}(\gamma_w(t), z) \text{ is } \lambda\text{-concave.} \quad (3.1.2)$$

But for most  $z \in B(p, \eta)$ , we can do much better. In fact, since  $\psi'' \ll -2$ ,

$$\psi \circ \text{dist}(\gamma_w(t), z) \text{ is } (-2)\text{-concave,} \quad (3.1.3)$$

unless  $\left| \angle(w, \uparrow_x^z) - \frac{\pi}{2} \right| \leq \tau \left( \frac{1}{|\psi''|} \middle| d \right)$ .

Next set

$$\log_x B(p, \eta) \equiv \{u \in T_x X \mid \gamma_u \text{ is a segment from } x \text{ to } \gamma_u(1) \in B(p, \eta)\}.$$

Then for some  $C > 0$  (that depends only on  $d$ ), we have

$$C \cdot \text{vol}(\log_x B(p, \eta)) \geq \text{vol}(B(p, \eta)) > v > 0. \quad (3.1.4)$$

Given  $w \in \Sigma_x$ , the set of “bad directions” for  $w$ ,

$$B(w) \equiv \left\{ u \in \Sigma_x \mid \left| \angle(w, u) - \frac{\pi}{2} \right| \leq \tau \left( \frac{1}{|\psi''|} \middle| d \right) \right\},$$

has  $(n-1)$ -dimensional volume

$$\text{vol}_{n-1}(B(w)) \leq \tau \left( \frac{1}{|\psi''|} \middle| d \right).$$

So

$$\text{vol}_n \left( \log_x B(p, \eta) \cap \left\{ u \in T_x X \mid \frac{u}{|u|} \in B(w) \right\} \right) \leq \tau \left( \frac{1}{|\psi''|} \middle| d \right) \tau(\eta),$$

and using Inequality (3.1.4),

$$\frac{\text{vol}_n \left( \log_x B(p, \eta) \cap \left\{ u \in T_x X \mid \frac{u}{|u|} \in B(w) \right\} \right)}{\text{vol}_n(\log_x B(p, \eta))} \leq \frac{C}{v} \tau \left( \frac{1}{|\psi''|} \middle| d \right) \tau(\eta).$$

By combining this with (3.1.2) and (3.1.3), we can force  $f_p$  to be strictly  $-1$ -concave on  $B(q, \delta)$  with appropriate choices of  $\psi$  and  $\delta$ .

Since  $\frac{1}{2} < \psi' \leq 2$  and  $\text{dist}(\cdot, z)$  is directionally differentiable and 1-Lipschitz, we apply the Bounded Convergence Theorem to differentiate under the integral and conclude that  $f_p$  is directionally differentiable and satisfies (3.1.1).  $\square$

A Gram-Schmidt argument as in [29] or [11] gives us the following.

**Lemma 3.2.** *Let*

$$p : U \longrightarrow \mathbb{R}^k$$

*be a submersion from an open subset  $U$  of a Riemannian manifold. Suppose that the component functions  $g_i$  of  $p$  are concave down and their gradients satisfy*

$$\angle(\nabla g_i, \nabla g_j) > \frac{\pi}{2}$$

*for all  $i \neq j$ . Let  $f : U \longrightarrow \mathbb{R}^k$  be a strictly concave down function so that for all  $i$ ,*

$$\angle(\nabla f, \nabla g_i) > \frac{\pi}{2}.$$

*Then the restrictions of  $f$  to the fibers of  $p$  are strictly concave down.*

In the context of a  $k$ -strained point, we combine the previous two results to get the following.

**Lemma 3.3.** *Let  $M_\alpha$  be a sequence of Riemannian  $n$ -manifolds with curvature  $\geq -1$  that converges to an  $n$ -dimensional Alexandrov space  $X$ . Suppose  $q \in X$  is  $(k, \delta, r)$ -strained by  $\{(a_i, b_i)\}_{i=1}^k$  and  $q_\alpha \in M_\alpha$  converge to  $q$ .*

*1. (cf [11], [13]) There is a convex neighborhood  $C$  of  $q$  and, for all but finitely many  $\alpha$ , convex neighborhoods  $C^\alpha$  of  $q_\alpha$  so that*

$$C^\alpha \longrightarrow C.$$

*2. For all but finitely many  $\alpha$ , there is a  $(\tau(\delta) + \tau(1/\alpha \mid r))$ -almost Riemannian submersion*

$$p^\alpha : C^\alpha \longrightarrow \mathbb{R}^k$$

*and a  $(-1)$ -concave function*

$$f_{C^\alpha}^\alpha : C^\alpha \longrightarrow \mathbb{R}$$

*so that the restriction of  $f_{C^\alpha}^\alpha$  to each fiber  $(p^\alpha)^{-1}(p^\alpha(x))$  of  $p^\alpha$  is strictly concave and has a unique interior maximum. Moreover,  $(\text{int}(C^\alpha), p^\alpha)$  is a vector bundle, and  $\text{int}(C^\alpha)$  is diffeomorphic to  $(0, 1)^n$  via a diffeomorphism  $\mu^\alpha$  that coincides with  $p^\alpha$  on the first  $k$  factors.*

*Proof.* We apply Proposition 1.14 and conclude that  $\Sigma_q X$  has a global  $(k, \tau(\delta))$ -strainer  $\{(v_i, w_i)\}_{i=1}^k$  so that

$$\frac{\pi}{2} < \text{dist}(v_i, v_j) \text{ for } i \neq j. \quad (3.3.1)$$

Moreover, for all  $\kappa \in (0, \frac{\pi}{4})$ , if  $\delta$  is sufficiently small compared to  $\kappa$ , there is a nonempty set  $E \subset \Sigma_q X$  so that for all  $e \in E$ ,

$$\frac{\pi}{2} < \text{dist}(e, v_i) < \frac{\pi}{2} + \kappa$$

and

$$\left| \text{dist}(e, w_i) - \frac{\pi}{2} \right| < \kappa.$$

Take  $E \subset \Sigma_q X$  to be the set of all directions that satisfy these inequalities.

By exponentiating approximations of these directions, it follows that there is a neighborhood  $N$  of  $q$  that is  $(k, \tau(\delta), \frac{r}{2})$ -strained by a strainer  $\{(a_i, b_i)\}_{i=1}^k$  that satisfies

$$\frac{\pi}{2} < \angle(\uparrow_x^{a_i}, \uparrow_x^{a_j}) \quad (3.3.2)$$

for all  $x \in N$  and  $i \neq j$ . Using Lemma 1.3, for some  $d > 0$ , we also have

$$\angle(a_i, q, \exp_q(de)) > \frac{\pi}{2}, \quad (3.3.3)$$

and

$$\left| \angle(b_i, q, \exp_q(de)) - \frac{\pi}{2} \right| < \tau(\delta, d, \kappa|r)$$

for all  $e \in E$  for which  $\exp_q(de)$  is defined. Since the last two inequalities are for comparison angles,  $q$  can be replaced by any  $x \in N$ , provided  $N$  is sufficiently small.

Let  $\{e_j\}$  be a  $\frac{\pi}{4}$ -net in  $E$  for which  $\exp_q(de_j)$  is defined. Apply the Perelman Concavity construction to  $\exp_q(de_j)$  and each of the strainer points to get strictly  $-1$ -concave functions  $\{f_{e_j}\}, \{g_{a_i}\}, \{g_{b_i}\}$  defined in a possibly smaller neighborhood  $U$  of  $q$ , and set

$$h = \min_{i,j} \{f_{e_j}, g_{a_i}, g_{b_i}\}.$$

For some  $\varepsilon > 0$ ,

$$\{\uparrow_q^{\tilde{a}_i}\} \cup \{\uparrow_q^{\tilde{b}_i}\} \cup \{\tilde{e}_j\} \text{ is a } \left(\frac{\pi}{2} - \varepsilon\right)\text{-net in } \Sigma_q X, \quad (3.3.4)$$

provided  $\tilde{a}_i, \tilde{b}_i$ , and  $\tilde{e}_j$  are sufficiently close to  $a_i, b_i$ , and  $e_j$ . By adding constants to the  $f_{e_j}$ s,  $g_{a_i}$ s, and  $g_{b_i}$ s, we can arrange that

$$f_{e_j}(q) = g_{a_i}(q) = g_{b_i}(q) \quad (3.3.5)$$

for all  $i$  and  $j$ . Combining (3.3.4) and (3.3.5) with the fact that  $h$  is strictly  $-1$ -concave on  $U$ , it follows that  $q$  is the unique maximum of  $h$  on  $U$ . Let  $C$  be a superlevel set of  $h$  that is contained in  $U$ .

Let  $M_\alpha$  be sufficiently close to  $X$ . The universality of Perelman's construction implies, in particular, that it is stable under Gromov-Hausdorff approximation, so each of  $h, C$ , and the  $f_{e_j}$ s,  $g_{a_i}$ s, and  $g_{b_i}$ s have approximations in  $M_\alpha$ . Call these approximations  $h^\alpha, C^\alpha, f_{e_j}^\alpha, g_{a_i}^\alpha$ , and  $g_{b_i}^\alpha$ . If  $\alpha$  is sufficiently large, the  $f_{e_j}^\alpha$ s,  $g_{a_i}^\alpha$ s, and  $g_{b_i}^\alpha$ s are strictly  $-1$ -concave,  $C^\alpha$  is convex, and the maximum of  $h^\alpha$  is in the interior of  $C^\alpha$ . So  $C^\alpha$  is diffeomorphic to an  $n$ -disk.

Set

$$\begin{aligned} p^\alpha & : C^\alpha \longrightarrow \mathbb{R}^k \\ p^\alpha & = (g_{a_1}^\alpha, g_{a_2}^\alpha, \dots, g_{a_k}^\alpha). \end{aligned}$$

Since  $C^\alpha$  is  $(k, \tau(\delta) + \tau(1/\alpha | r), r)$ -strained, it follows from Lemma 1.4 that  $p^\alpha$  is a  $(\tau(\delta) + \tau(1/\alpha | r))$ -almost Riemannian submersion.

Proposition 2.13 and inequalities (3.3.2) and (3.3.3) give us

$$\begin{aligned} \angle(\nabla g_{a_i}^\alpha, \nabla g_{a_j}^\alpha) & > \frac{\pi}{2} \text{ and} \\ \angle(\nabla g_{a_i}^\alpha, \nabla f_{e_j}^\alpha) & > \frac{\pi}{2}, \end{aligned} \quad (3.3.6)$$

for  $\alpha$  sufficiently large. Combining this with Lemma 3.2, it follows that the restriction of each  $f_{e_j}^\alpha$  to the fibers of  $p^\alpha$  is concave down. Set

$$f_{C^\alpha}^\alpha \equiv \min_i \{f_{e_i}^\alpha\}.$$

It follows that the restriction of  $f_C^\alpha$  to each fiber  $(p^\alpha)^{-1}(p^\alpha(x))$  of  $p^\alpha$  is strictly concave, and, after possibly restricting the base of  $p^\alpha$ , that each  $f_C^\alpha|_{(p^\alpha)^{-1}(p^\alpha(x))}$  has a unique interior maximum. In particular, each fiber of  $p^\alpha$  is a disk, so there is a diffeomorphism  $\mu^\alpha : C^\alpha \rightarrow I^n$  whose first  $k$  coordinate functions are  $p^\alpha = (g_{a_1}^\alpha, g_{a_2}^\alpha, \dots, g_{a_k}^\alpha)$ .

To see that  $(C^\alpha, p^\alpha)$  is a vector bundle, let  $s_x^\alpha$  be the unique maximum of  $f_{C^\alpha}^\alpha$  restricted to  $(p^\alpha)^{-1}(p^\alpha(x))$ . The collection

$$S^\alpha \equiv \{s_x^\alpha\}_{x \in C_j^\alpha}$$

forms a  $\dim(S)$ -dimensional submanifold of  $C^\alpha$ . The gradients of  $f^\alpha$  restricted to the fibers of  $p^\alpha$  allow us to identify the fibers of  $p^\alpha$  with the normal bundle of  $S^\alpha$ , thus giving  $(C^\alpha, p^\alpha)$  the structure of a trivial vector bundle.  $\square$

Recall that in Theorem 2.14 we construct a cover of  $X$  by subcollections,  $\mathcal{O}^X \equiv \{B_j^X(\rho^X)\}_j$  and  $\{\mathcal{O}^{S_i}\}_i \equiv \{\{B_j^{S_i}(\rho^{S_i})\}_j\}_i$ . To simplify notation, we will refer to a  $B_j^{S_i}(\rho^{S_i})$  or to a  $B_j^X(\rho^X)$  as simply  $B_j(\rho_j)$ , and let  $p_j$  be the map  $B_j(\rho_j) \rightarrow \mathbb{R}^{\dim(S_i)}$  from (2.13.1). We write  $S_j$  for the element of  $\mathcal{S}^{\text{ext}}$  associated to  $B_j(\rho_j)$ . Thus for  $S \in \mathcal{S}$  and  $B_j(\rho_j) \in \mathcal{O}^S$ , we have  $S_j = S$ . Of course,  $S_j$  might be our top stratum,  $(X \setminus \cup_{S \in \mathcal{S}} S)$ , and, with this notation, many of the  $S_j$ s are likely to be equal to each other.

**Theorem 3.4.** *Let  $X$  and  $\{M_\alpha\}_\alpha$  be as in the TNST. Given  $\varepsilon > 0$ , let  $\{B_j(\rho_j)\}_j$  be the open cover of  $X$  from Theorem 2.14. If the  $\rho_j$ s are sufficiently small, then the following hold.*

1. *For all but finitely many  $\alpha$  and for all  $j$  for which  $S_j$  is not the top stratum, there is a  $3\rho_j$ -ball  $B_j^\alpha(3\rho_j) \subset M_\alpha$  so that*

$$B_j^\alpha(3\rho_j) \longrightarrow B_j(3\rho_j)$$

*as  $\alpha \rightarrow \infty$ . Moreover, there are  $\varepsilon$ -almost Riemannian submersions*

$$\begin{aligned} p_j^\alpha &: B_j^\alpha(3\rho_j) \longrightarrow \mathbb{R}^{\dim S_j} \\ \mu_j &: B_j(3\rho_j) \cap S_j \longrightarrow \mathbb{R}^{\dim S_j} \end{aligned}$$

*so that the  $\mu_j$ s are embeddings, the  $p_j^\alpha$ s are restrictions of projection maps of vector bundles, and*

$$p_j^\alpha \longrightarrow p_j$$

*as  $\alpha \rightarrow \infty$ .*

2. *Part 5 of Theorem 2.14 holds with the balls of  $X$  replaced by the corresponding balls of  $M^\alpha$  and with  $p_k^N$  and  $p_{j(k)}^S$  replaced with the corresponding maps from Part 1.*

3. *If  $S_j$  is the top stratum, then Parts 1 and 2 hold except that the  $p_j^\alpha$ s are embeddings that are  $\tau(\delta)$ -almost Riemannian submersions rather than  $\varepsilon$ -almost Riemannian submersions.*

**Remark 3.5.** *Since  $p_j^\alpha$  is an embedding when  $S_j$  is the top stratum, we will write  $\mu_j^\alpha$  for  $p_j^\alpha$  in this case.*



*Proof.* We apply Lemma 3.3 to the center of each ball of the open cover of Theorem 2.14. By Lemma 3.3, if  $\rho$  is sufficiently small, then each  $B_j(3\rho_j)$  is contained in a convex set  $C_j$  of  $X$ , and for each  $j$  and all but finitely many  $\alpha$ , there is a convex set  $C_j^\alpha$  with

$$C_j^\alpha \longrightarrow C_j.$$

For each  $j$  and all but finitely many  $\alpha$ , Part 2 of Lemma 3.3 and its proof give us

$$p_j^\alpha : C_j^\alpha \longrightarrow \mathbb{R}^{\dim S_j} \text{ and } p_j : C_j \longrightarrow \mathbb{R}^{\dim S_j} \text{ with } p_j^\alpha \longrightarrow p_j \text{ as } \alpha \rightarrow \infty.$$

By defining  $\mu_j \equiv p_j|_S$ , we have the desired maps. If  $S_j$  is not the top stratum, then it follows from Part 2 of Lemma 3.3 that  $p_j^\alpha$  and  $\mu_j$  are  $\tau(\tilde{\delta}) + \tau(1/\alpha |r)$ -almost Riemannian submersions. Since  $\tilde{\delta}$  and  $1/\alpha$  can be arbitrarily small, we may arrange that  $p_j^\alpha$  and  $\mu_j$  are  $\varepsilon$ -almost Riemannian submersions. By the proof of Theorem 5.4 of [1], the  $\mu_j$ s are embeddings, provided  $\rho_j$  is also sufficiently small, and Part 1 follows.

Part 2, that is,

$$\begin{aligned} B_k^{N,\alpha}(3\rho^N) &\subseteq B_{j(k)}^{S_{j(k)},\alpha}(\rho^S) \text{ and} \\ \pi_{\dim(S_{j(k)})} \circ p_k^{N,\alpha} &= p_{j(k)}^{S_{j(k)},\alpha}, \end{aligned}$$

then follows from Part 5 of Theorem 2.14 and the construction of the  $p_k^{N,\alpha}$ s and the  $p_{j(k)}^{S_{j(k)},\alpha}$ s.

The proof of Part 3 is the same, except that we have not assumed that the top stratum is a Riemannian manifold. Rather we have only assumed that every point in the top stratum is  $(n, \delta)$ -strained. Thus  $\delta$  cannot be taken to be arbitrarily small, and we can only conclude, using Lemma 1.4, that  $p_j^\alpha$  and  $\mu_j$  are  $\tau(\delta)$ -almost Riemannian submersions.  $\square$

**Remark.** In the proof of Part 1 of the previous result, we exploited the fact that both  $\frac{1}{\alpha}$  and the quantity  $\tilde{\delta}$  from Corollary 2.10 can be arbitrarily small. Using this we replaced each of  $\tau(\frac{1}{\alpha}|r)$ ,  $\tau(\tilde{\delta})$ , and  $\tau(\frac{1}{\alpha}|r) + \tau(\tilde{\delta})$  by an arbitrarily small positive number  $\varepsilon$ . For similar reasons, we replaced  $\tau(\frac{1}{\alpha}|\rho, r) + \tau(\delta)$  with  $\tau(\delta)$  in the proof of Part 3. The quantities  $\tau(\frac{1}{\alpha}|\rho, r)$  and  $\tau(\frac{1}{\alpha}|r)$  will appear in the sequel, but only when they are needed to clarify a link between results that appear prior to and subsequent to this remark. Whenever such a clarification is not needed, to simplify notation, we will make the substitutions of the previous proof, that is,

$$\begin{aligned} \tau\left(\frac{1}{\alpha}|r\right) + \tau(\tilde{\delta}) &\text{ is replaced by } \varepsilon, \text{ and} \\ \tau\left(\frac{1}{\alpha}|\rho, r\right) + \tau(\delta) &\text{ is replaced by } \tau(\delta) \end{aligned}$$

For the remainder of the paper,  $\varepsilon$  is the number from Theorem 2.14.

#### 4. SUBMERSIONS OF NEARBY CONVEX SETS

In this section, we prove Proposition 4.2, which says that the submersions of Theorem 3.4 are  $C^1$ -close on their overlaps. We then prove the analogous result for the top stratum in

Proposition 4.3 (below). Ultimately, these results will allow us to glue the locally defined maps together via Theorem 5.3.

We start by showing that the submersions of neighboring balls have nearly the same horizontal spaces.

**Lemma 4.1.** *Let  $X$  and  $\{M_\alpha\}_\alpha$  be as in the TNST. For  $S \in \mathcal{S}$ , let*

$$\begin{aligned} p_s^\alpha &: B_s^\alpha(3\rho) \longrightarrow \mathbb{R}^{\dim S} \text{ and} \\ p_t^\alpha &: B_t^\alpha(3\rho) \longrightarrow \mathbb{R}^{\dim S} \end{aligned}$$

*be two of the  $\varepsilon$ -almost Riemannian submersions from Part 1 of Theorem 3.4. At all points of  $B_s^\alpha(3\rho) \cap B_t^\alpha(3\rho)$ , the unit spheres in the horizontal spaces of  $p_s^\alpha$  and  $p_t^\alpha$  are within  $\tau(\varepsilon)$  of each other.*

*Proof.* Let the  $(\dim(S), \tilde{\delta}, r)$ -strainers of  $B_s(3\rho)$  and  $B_t(3\rho)$  be  $\{(a_i, b_i)\}_{i=1}^{\dim S}$  and  $\{(c_i, d_i)\}_{i=1}^{\dim S}$ , respectively. Let  $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{\dim S}$  and  $\{(c_i^\alpha, d_i^\alpha)\}_{i=1}^{\dim S}$  converge to  $\{(a_i, b_i)\}_{i=1}^{\dim S}$  and  $\{(c_i, d_i)\}_{i=1}^{\dim S}$ . By considering the formula for orthogonal projection with respect to an orthonormal basis, we see that it suffices to show that for  $y^\alpha \in B_s^\alpha(3\rho) \cap B_t^\alpha(3\rho)$ ,

$$\left| \det \left( \cos \angle \left( \uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c_j^\alpha} \right) \right)_{i,j} - 1 \right| < \varepsilon. \quad (4.1.1)$$

By Proposition 1.5,

$$\left| \angle \left( \uparrow_{y^\alpha}^{a_i^\alpha}, \uparrow_{y^\alpha}^{c_j^\alpha} \right) - \angle \left( \uparrow_y^{a_i}, \uparrow_y^{c_j} \right) \right| < \varepsilon. \quad (4.1.2)$$

On the other hand, by Inequality (2.10.1), both  $\{\uparrow_y^{a_i}\}_{i=1}^{\dim(S)}$  and  $\{\uparrow_y^{c_j}\}_{j=1}^{\dim(S)}$  are within  $\varepsilon$  of  $T_y S$ , so

$$\left| \det \left( \cos \angle \left( \uparrow_y^{a_i}, \uparrow_y^{c_j} \right) \right)_{i,j} - 1 \right| < \tau(\varepsilon).$$

The result follows by combining the previous two displays.  $\square$

**Proposition 4.2.** *Let  $X$  and  $\{M_\alpha\}_\alpha$  be as in the TNST. For  $S \in \mathcal{S}$ , let  $\mathcal{O}^S$  be as in Theorem 2.14. Let  $B(S, 2\nu)$  be the  $2\nu$ -neighborhood of  $S$  with respect to a fixed metric on  $(\Pi_\alpha M_\alpha) \amalg X$  that realizes the Gromov-Hausdorff convergence. Let*

$$\begin{aligned} p_j^\alpha &: B_j^\alpha(3\rho) \longrightarrow \mathbb{R}^{\dim S} \\ \mu_j &: B_j(3\rho) \cap S \longrightarrow \mathbb{R}^{\dim S} \end{aligned}$$

*be the  $\varepsilon$ -almost Riemannian submersions from Theorem 3.4.*

*Then on  $B_j^\alpha(3\rho) \cap B_k^\alpha(3\rho) \cap B(S, 2\nu)$ ,*

$$|p_k^\alpha - \mu_k \circ \mu_j^{-1} \circ p_j^\alpha|_{C^0} \leq \tau \left( \frac{1}{\alpha}, \nu \right), \quad (4.2.1)$$

*and*

$$|p_k^\alpha - \mu_k \circ \mu_j^{-1} \circ p_j^\alpha|_{C^1} \leq \tau(\varepsilon). \quad (4.2.2)$$

*Proof.* Suppose  $y \in B_j(3\rho) \cap B_k(3\rho) \cap B(S, 2\nu)$ ,  $y^\alpha \in B_j^\alpha(3\rho) \cap B_k^\alpha(3\rho)$ , and  $y^\alpha \rightarrow y$ . Then

$$\text{dist}(\mu_j^{-1} \circ p_j^\alpha(y^\alpha), y) < \tau\left(\frac{1}{\alpha}, \nu\right)$$

and

$$\text{dist}(\mu_k^{-1} \circ p_k^\alpha(y^\alpha), y) < \tau\left(\frac{1}{\alpha}, \nu\right),$$

so

$$\text{dist}(\mu_j^{-1} \circ p_j^\alpha(y^\alpha), \mu_k^{-1} \circ p_k^\alpha(y^\alpha)) < \tau\left(\frac{1}{\alpha}, \nu\right).$$

Since  $\mu_k$  is  $(1 + \varepsilon)$ -bilipschitz, Inequality (4.2.1) follows from the previous display.

To make the proof of Inequality (4.2.2) easier to follow, we change the indices “ $j$ ” and “ $k$ ” to “ $a$ ” and “ $c$ ”, and prove (4.2.2) for submersions  $p_a^\alpha$  and  $p_c^\alpha$  and embeddings  $\mu_a$  and  $\mu_c$ , whose defining strainers are  $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n$ ,  $\{(c_i^\alpha, d_i^\alpha)\}_{i=1}^n$ ,  $\{(a_i, b_i)\}_{i=1}^n$ , and  $\{(c_i, d_i)\}_{i=1}^n$ , respectively.

We suppose that for all  $i$ ,

$$\begin{aligned} \text{dist}(a_i, a_i^\alpha) &< \varepsilon, \text{dist}(b_i, b_i^\alpha) < \varepsilon, \\ \text{dist}(c_i, c_i^\alpha) &< \varepsilon, \text{ and } \text{dist}(d_i, d_i^\alpha) < \varepsilon. \end{aligned}$$

Let  $x^\alpha$  be any point in the domains of  $p_a^\alpha$  and  $p_c^\alpha$ . Let  $x \in X$  satisfy  $\text{dist}(x, x^\alpha) < \varepsilon$ .

Inequalities (4.1.1) and (4.1.2) give us the hypotheses of Proposition 1.6. Thus given a unit

$$Y^\alpha \in \text{span} \left\{ \uparrow_{x^\alpha}^{a_i^\alpha} \right\}_{i=1}^{\dim S},$$

there is a  $Y \in T_x S$  so that for all  $i$ ,

$$\left| \angle(Y, \uparrow_x^{a_i}) - \angle(Y^\alpha, \uparrow_{x^\alpha}^{a_i^\alpha}) \right| < \tau(\varepsilon) \quad (4.2.3)$$

and

$$\left| \angle(Y, \uparrow_x^{c_i}) - \angle(Y^\alpha, \uparrow_{x^\alpha}^{c_i^\alpha}) \right| < \tau(\varepsilon). \quad (4.2.4)$$

Inequality (4.2.3) gives us

$$|D(\mu_a)_x(Y) - D(p_a^\alpha)_{x^\alpha}(Y^\alpha)| < \tau(\varepsilon), \quad (4.2.5)$$

and Inequality (4.2.4) gives us

$$|D(\mu_c)_x(Y) - D(p_c^\alpha)_{x^\alpha}(Y^\alpha)| < \tau(\varepsilon).$$

Since  $D(\mu_c \circ \mu_a^{-1})$  is  $\left(1 + \tau(\tilde{\delta})\right)$ -bilipschitz, Inequality (4.2.5) gives us

$$|D(\mu_c)_x(Y) - D(\mu_c \circ \mu_a^{-1} \circ p_a^\alpha)_{x^\alpha}(Y^\alpha)| < \tau(\varepsilon).$$

Inequality (4.2.2) follows by combining the previous two displays.  $\square$

For the top stratum the analogous result is

**Proposition 4.3.** *Let  $X$  and  $\{M_\alpha\}_{\alpha \in \mathbb{N}}$  be as in Theorem B. Let  $\mathcal{O}^X = \{B_j(3\rho)\}_j$  be as in Theorem 2.14. The  $\tau(\delta)$ -almost Riemannian submersions*

$$\mu_j^\alpha : B_j^\alpha(3\rho) \longrightarrow \mathbb{R}^n$$

*of Part 3 of Theorem 3.4 have the following property.*

*For  $\beta, \sigma \in \mathbb{N}$  with  $\sigma \leq \beta$  and for all  $j, k$ ,*

$$\left| \mu_k^\sigma - \mu_k^\beta \circ \left( \mu_j^\beta \right)^{-1} \circ \mu_j^\sigma \right|_{C^1} \leq \tau(\delta) \quad (4.3.1)$$

*and*

$$\left| \mu_k^\sigma - \mu_k^\beta \circ \left( \mu_j^\beta \right)^{-1} \circ \mu_j^\sigma \right|_{C^0} \leq \tau \left( \frac{1}{\sigma} \mid r \right) \quad (4.3.2)$$

*on  $B_j^\sigma(3\rho) \cap B_k^\sigma(3\rho)$ .*

*Proof.* Suppose  $y \in B_j(3\rho) \cap B_k(3\rho)$ ,  $y^\sigma \in B_j^\sigma(3\rho) \cap B_k^\sigma(3\rho)$ ,  $y^\beta \in B_j^\beta(3\rho) \cap B_k^\beta(3\rho)$ ,  $\text{dist}(y^\sigma, y) < \tau \left( \frac{1}{\sigma} \mid r \right)$ , and  $\text{dist}(y^\beta, y) < \tau \left( \frac{1}{\sigma} \mid r \right)$ . Then

$$\left| \mu_j^\beta(y^\beta) - \mu_j^\sigma(y^\sigma) \right| < \tau \left( \frac{1}{\sigma} \mid r \right) \text{ and} \quad (4.3.3)$$

$$\left| \mu_k^\beta(y^\beta) - \mu_k^\sigma(y^\sigma) \right| < \tau \left( \frac{1}{\sigma} \mid r \right). \quad (4.3.4)$$

Since  $\mu_k^\beta \circ \left( \mu_j^\beta \right)^{-1}$  is  $(1 + \tau(\delta))$ -Lipschitz, Inequality (4.3.3) gives

$$\left| \mu_k^\beta(y^\beta) - \mu_k^\beta \circ \left( \mu_j^\beta \right)^{-1} \circ \mu_j^\sigma(y^\sigma) \right| < \tau \left( \frac{1}{\sigma} \mid r \right),$$

which, together with Inequality (4.3.4), gives Inequality (4.3.2).

Suppose  $M, \tilde{M} \in \{M_\alpha\}_{\alpha \geq \sigma}$ . To make the proof of Inequality (4.3.1) easier to follow, we change the indices “ $j$ ” and “ $k$ ” to “ $a$ ” and “ $c$ ”, and prove (4.3.1) for coordinate charts  $\mu_a$  and  $\mu_c$  of  $M$  and  $\tilde{\mu}_a$  and  $\tilde{\mu}_c$  of  $\tilde{M}$ , whose defining strainers are  $\{(a_i, b_i)\}_{i=1}^n$ ,  $\{(c_i, d_i)\}_{i=1}^n$ ,  $\{(\tilde{a}_i, \tilde{b}_i)\}_{i=1}^n$ , and  $\{(\tilde{c}_i, \tilde{d}_i)\}_{i=1}^n$ , respectively.

Suppose that for all  $i$ ,

$$\begin{aligned} \text{dist}(a_i, \tilde{a}_i) &< \tau \left( \frac{1}{\sigma} \mid r \right), \text{dist}(b_i, \tilde{b}_i) < \tau \left( \frac{1}{\sigma} \mid r \right), \\ \text{dist}(c_i, \tilde{c}_i) &< \tau \left( \frac{1}{\sigma} \mid r \right), \text{and dist}(d_i, \tilde{d}_i) < \tau \left( \frac{1}{\sigma} \mid r \right). \end{aligned} \quad (4.3.5)$$

Suppose also that  $y \in M$  is in the domains of both  $\mu_a$  and  $\mu_c$ , that  $\tilde{y} \in \tilde{M}$  is in the domains of both  $\tilde{\mu}_a$  and  $\tilde{\mu}_c$ , and that  $\text{dist}(y, \tilde{y}) < \tau \left( \frac{1}{\sigma} \mid r \right)$ .

Proposition 1.5 and the inequalities in (4.3.5) give us the hypotheses of Proposition 1.6. So given a unit

$$Y \in \Sigma_y,$$

there is a unit

$$\tilde{Y} \in \Sigma_{\tilde{y}}$$

so that for all  $i$ ,

$$\left| \triangleleft (Y, \uparrow_y^{a_i}) - \triangleleft (\tilde{Y}, \uparrow_{\tilde{y}}^{\tilde{a}_i}) \right| < \tau \left( \frac{1}{\sigma} |r| \right) + \tau(\delta)$$

and

$$\left| \triangleleft (Y, \uparrow_y^{c_i}) - \triangleleft (\tilde{Y}, \uparrow_{\tilde{y}}^{\tilde{c}_i}) \right| < \tau \left( \frac{1}{\sigma} |r| \right) + \tau(\delta).$$

Combining this with the definitions of the  $\mu$ s,

$$\left| D\tilde{\mu}_a(\tilde{Y}) - D\mu_a(Y) \right| \leq \tau(\delta) + \tau \left( \frac{1}{\sigma} |r| \right) \quad (4.3.6)$$

and

$$\left| D\tilde{\mu}_c(\tilde{Y}) - D\mu_c(Y) \right| \leq \tau(\delta) + \tau \left( \frac{1}{\sigma} |r| \right). \quad (4.3.7)$$

Since  $D(\tilde{\mu}_c \circ \tilde{\mu}_a^{-1})$  is  $(1 + \tau(\delta))$ -bilipschitz, Inequality (4.3.6) gives

$$\left| (D\tilde{\mu}_c)(\tilde{Y}) - D(\tilde{\mu}_c \circ \tilde{\mu}_a^{-1} \circ \mu_a)(Y) \right| \leq \tau(\delta) + \tau \left( \frac{1}{\sigma} |r| \right).$$

Combined with Inequality (4.3.7), this gives

$$\left| D\mu_c(Y) - D(\tilde{\mu}_c \circ \tilde{\mu}_a^{-1} \circ \mu_a)(Y) \right| \leq \tau(\delta) + \tau \left( \frac{1}{\sigma} |r| \right).$$

Inequality (4.3.1) follows by recalling that  $\tau \left( \frac{1}{\sigma} |r| \right)$  can be arbitrarily small.  $\square$

## 5. GLUING $C^1$ -CLOSE SUBMERSIONS

In this section we state Theorem 5.3, an abstract gluing theorem for submersions, which, together with Proposition 4.2, will allow us to glue together the locally defined submersions of Theorem 3.4. It is based on the principle that a space of submersions is locally contractible in the  $C^1$ -topology. Since there are somewhat similar results elsewhere in the literature (cf [3], [14], [20]), we defer the proof of Theorem 5.3 to the appendix (8). Before stating Theorem 5.3, we establish some background definitions and hypotheses.

**Definition 5.1.** *We say that two collections of sets  $\{C_i\}_{i \in I}$  and  $\{T_i\}_{i \in I}$  have the same intersection pattern provided  $C_i \cap C_j \neq \emptyset$  if and only  $T_i \cap T_j \neq \emptyset$ .*

**Definition 5.2.** *If  $\mathcal{C} \equiv \{C_i\}_{i \in I}$  is a collection of subsets of a space  $X$ , we let  $\text{cl}(\mathcal{C}) \equiv \{\bar{C}_i\}_{i \in I}$  be the collection of their closures.*

Throughout this section, we assume the following:

1. The collection  $\tilde{\mathcal{C}} \equiv \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l}$  of  $\rho$ -balls in the Riemannian  $n$ -manifold  $M$  has order  $\leq \mathfrak{o}$  and satisfies  $\text{dist} \left( \overline{\tilde{B}_i(\rho)}, \overline{\tilde{B}_i(3\rho) \setminus \tilde{B}_i(2\rho)} \right) = \rho$ .
2. For  $\eta \in (0, 1)$  and  $l \geq 1$ ,

$$\tilde{p}_i : \tilde{B}_i(3\rho) \longrightarrow \mathbb{R}^l$$

are  $\eta$ -almost Riemannian submersions.

3.  $\mathcal{C} = \{B_i(\rho)\}_{i=1}^{m_l}$  is a collection of  $\rho$ -balls in a Riemannian  $l$ -manifold  $S$ .
4. There are coordinate charts

$$\mu_i : B_i(3\rho) \longrightarrow \mathbb{R}^l$$

that are  $\eta$ -almost Riemannian submersions.

5. The collections  $\mathcal{C}$ ,  $\tilde{\mathcal{C}}$ ,  $\text{cl}(\mathcal{C})$ , and  $\text{cl}(\tilde{\mathcal{C}})$  have the same intersection pattern.

**Theorem 5.3.** (*Submersion Gluing Theorem*) Assume that  $M$  and  $S$  satisfy Hypotheses 1–5, above.

There are  $\xi_0(\mathfrak{o}, l) > 0$ ,  $\eta(l) > 0$ , and  $\varepsilon_0(l) > 0$  with the following property: Suppose that for all  $i$ ,

$$\text{dist}_{\text{Haus}}\left(\tilde{p}_i\left(\tilde{B}_i(\rho)\right), \mu_i\left(B_i(\rho)\right)\right) < \xi_0, \quad (5.3.1)$$

and, for all pairs  $(i, j)$ ,

$$\left|\tilde{p}_i - \mu_i \circ \mu_j^{-1} \circ \tilde{p}_j\right|_{C^0} < \xi \leq \xi_0 \quad (5.3.2)$$

and

$$\left|\tilde{p}_i - \mu_i \circ \mu_j^{-1} \circ \tilde{p}_j\right|_{C^1} < \varepsilon \leq \varepsilon_0 \quad (5.3.3)$$

on  $\tilde{B}_i(3\rho) \cap \tilde{B}_j(3\rho)$ .

Then there is a submersion  $P : \cup_{i=1}^{m_l} \tilde{B}_i(\rho) \longrightarrow P\left(\cup_{i=1}^{m_l} \tilde{B}_i(\rho)\right) \subset S$  so that

$$P|_{\tilde{B}_{m_l}(\rho)} = \mu_{m_l}^{-1} \circ \tilde{p}_{m_l}, \quad (5.3.4)$$

and, on each  $\tilde{B}_i(\rho)$ ,

$$\left|\mu_i \circ P - \tilde{p}_i\right|_{C^0} < \tau(\xi) \quad (5.3.5)$$

and

$$\left|\mu_i \circ P - \tilde{p}_i\right|_{C^1} < \tau(\varepsilon) + \tau(\xi|\rho). \quad (5.3.6)$$

**Remark 5.4.** In the proof of Theorem 5.3, we show that the functions  $\tau$  on the righthand sides of Inequalities (5.3.5) and (5.3.6) can be taken to be

$$\tau(\xi) = (1 + \eta)^{2\mathfrak{o}} \xi \text{ and}$$

$$\tau(\varepsilon) + \tau(\xi|\rho) = (1 + \eta)^{2(\mathfrak{o}-1)} \varepsilon + \frac{2}{\rho} \xi (\mathfrak{o} - 1) (1 + \eta)^{2(\mathfrak{o}-1)}.$$

The reader might be more comfortable calling these functions  $\tau(\xi|\eta, \mathfrak{o})$  and  $\tau(\varepsilon, \eta|\mathfrak{o}) + \tau(\xi|\eta, \mathfrak{o}, \rho)$ . In our applications,  $\eta$  is small,  $\xi \ll \eta$ , and  $\mathfrak{o}$  is a fixed constant that only depends on  $X$ , so for simpler notation, we have chosen to write them as in Theorem 5.3.

While Theorem 5.3 is the main abstract gluing tool used to construct the bundle maps of the TNST, we will also need the following corollary to establish Properties 4 and 5 of the TNST.

**Corollary 5.5.** *Let  $M$ ,  $N$ , and  $S$  be compact Riemannian manifolds of dimensions  $n \geq k \geq l$ , respectively. Suppose the hypotheses of Theorem 5.3 hold for  $M$  and  $S$ , and that for some  $\rho_R > 0$ ,  $\{B_i(\rho_R)\}_{i=1}^{m_R}$  is a collection of  $\rho_R$  balls in  $M$ , so that*

$$\begin{aligned} \text{dist} \left( \overline{B_i(\rho_R)}, \overline{B_i(3\rho_R) \setminus B_i(2\rho_R)} \right) &= \rho_R \text{ and} \\ \cup_{i \in I_R} B_i(3\rho_R) &\subset \cup_{i=1}^{m_l} \tilde{B}_i(\rho), \end{aligned}$$

where  $I_R$  is some subset of  $\{1, 2, \dots, m_R\}$  for which the order of  $\{B_i(3\rho_R)\}_{i \in I_R}$  is  $\leq \mathfrak{o}$ . Then there are  $\xi_0(l, k, \mathfrak{o}) > 0$ ,  $\eta(l, k) > 0$ , and  $\varepsilon_0(l, k) > 0$  with the following property.

Suppose that

$$\begin{aligned} R &: \cup_{i=1}^{m_R} B_i(3\rho_R) \longrightarrow N \text{ and} \\ Q &: N \longrightarrow S \end{aligned}$$

are  $\eta$ -almost Riemannian submersions so that for each  $i = 1, 2, \dots, m_l$ , on  $\cup_{i \in I_R} B_i(3\rho_R) \cap \tilde{B}_i(3\rho)$ , we have

$$|\tilde{p}_i - \mu_i \circ Q \circ R|_{C^0} < \xi \leq \xi_0 \quad (5.5.1)$$

and

$$|\tilde{p}_i - \mu_i \circ Q \circ R|_{C^1} < \varepsilon \leq \varepsilon_0. \quad (5.5.2)$$

Then there is a submersion  $P : \cup_{i=1}^{m_l} \tilde{B}_i(\rho) \longrightarrow P(\cup_{i=1}^{m_l} \tilde{B}_i(\rho)) \subset S$  so that on  $\cup_{i \in I_R} B_i(\rho_R)$

$$P = Q \circ R, \quad (5.5.3)$$

and, on each  $\tilde{B}_i(\rho)$ ,

$$|\mu_i \circ P - \tilde{p}_i|_{C^0} < \tau(\xi) \quad (5.5.4)$$

and

$$|\mu_i \circ P - \tilde{p}_i|_{C^1} < \tau(\varepsilon) + \tau(\xi|\rho). \quad (5.5.5)$$

Since Theorem 5.3 and Corollary 5.5 are similar to other results in the literature, we defer their proofs to the appendix (8).

## 6. ESTABLISHING THE TUBULAR NEIGHBORHOOD STABILITY THEOREM

In this section, we prove Parts 1—5 of the TNST. Parts 1, 2 and 5 are established in Subsection 6.1 and Parts 3 and 4 in Subsection 6.3. Part 6 of the TNST is proven in Section 7.

**6.1. The Vector Bundles of the TNST.** Part 2 of the TNST is a consequence of the following result.

**Proposition 6.2.** *Let  $X$  and  $\{M_\alpha\}_\alpha$  be as in the TNST. Given  $\varepsilon > 0$ , let  $\{B_j(\rho_j)\}_j$  be the open cover of  $X$  from Theorem 2.14, and let*

$$\begin{aligned} p_j^\alpha &: B_j^\alpha(3\rho_j) \longrightarrow \mathbb{R}^{\dim S_j} \text{ and} \\ \mu_j &: B_j(3\rho_j) \cap S_j \longrightarrow \mathbb{R}^{\dim S_j} \end{aligned}$$

be the  $\varepsilon$ -almost Riemannian submersions from Theorem 3.4. For  $S_j \in \mathcal{S}$ , let  $O_j \subset S_j$  be as in Part 1 of Theorem 2.14. If  $\frac{1}{\alpha}$  and the  $\rho_j$ s are sufficiently small, then:

1. There is an open  $\mathcal{U}_\alpha^j \subset M_\alpha$  and a surjective  $C^1$ -vector bundle

$$P_\alpha^{S_j} : \mathcal{U}_\alpha^j \longrightarrow O_j \subset S_j$$

which is also an  $\varepsilon$ -almost Riemannian submersion.

2. The sets  $\mathcal{U}_\alpha^j$  from Part 1 satisfy

$$\text{dist}_{GH}(\mathcal{U}_\alpha^j, O_j) < \tau\left(\frac{1}{\alpha}, \nu\right),$$

where  $\nu$  is as in Proposition 4.2.

3. For  $S \in \mathcal{S}$  and  $j$  such that  $B_j(\rho_j) \in \mathcal{O}^S$ ,

$$|\mu_j \circ P_\alpha^S - p_j^\alpha|_{C^1} < \tau(\varepsilon) \quad (6.2.1)$$

on  $B_j^\alpha(\rho) \cap \mathcal{U}_\alpha$ .

*Proof.* Since the entire result is about a single  $S_j \in \mathcal{S}$ , for simplicity we write  $S$  for  $S_j$ .

By combining Proposition 4.2 with Theorems 3.4 and 5.3, we get the existence of  $\tilde{\mathcal{U}}_\alpha \subset M_\alpha$  with

$$\text{dist}_{GH}(\tilde{\mathcal{U}}_\alpha, O) < \tau\left(\frac{1}{\alpha}, \nu\right)$$

and a  $\tau(\varepsilon)$ -almost Riemannian submersion

$$P_\alpha : \tilde{\mathcal{U}}_\alpha \longrightarrow O \subset S$$

that satisfies Equation (6.2.1).

By the Stability Theorem ([20, 14]), for all but finitely many  $\alpha$ , there is a  $\tau(1/\alpha)$ -homeomorphism

$$h_\alpha : X \longrightarrow M_\alpha.$$

Set  $S_\alpha = h_\alpha(S)$ . Since the conclusion of Lemma 2.11 is Gromov-Hausdorff stable, given any  $\varepsilon > 0$ , there is a  $\nu > 0$  and a unit vector field  $V$  on  $(B(S, 2\nu) \setminus B(S, \nu)) \cap B_j^\alpha(\rho)$  with

$$D_V \text{dist}(S_\alpha, \cdot) > 1 - \varepsilon \text{ and} \quad (6.2.2)$$

$$|Dp_j^\alpha(V)| < \tau(\tilde{\delta}) + \tau(\rho|r). \quad (6.2.3)$$

Since the Riemannian convolution method of [9] preserves regularity, it follows from (6.2.2) that for an appropriate convolution  $d_\alpha$ ,

$$D_V d_\alpha > 1 - \varepsilon$$

on  $(B(S, 2\nu) \setminus B(S, \nu)) \cap B_j^\alpha(\rho)$ , where  $d_\alpha$  is as close as we please to  $\text{dist}(S_\alpha, \cdot)$  in the  $C^0$ -topology.

It follows from Inequalities (4.2.1), (4.2.2), (5.3.6), and (6.2.3) that  $V$  is almost vertical for  $P_\alpha$  on  $\tilde{\mathcal{U}}_\alpha \cap (d_\alpha^{-1}([0, 2\nu]) \setminus d_\alpha^{-1}([0, \nu]))$ , that is,

$$|DP_\alpha(V)| < \tau(\tilde{\delta}) + \tau\left(\frac{1}{\alpha}, \nu|\rho\right) + \tau(\rho|r).$$

Thus the restriction of  $P_\alpha$  to  $\tilde{\mathcal{U}}_\alpha \cap d_\alpha^{-1}([0, 2\nu])$  is a submersion. Since a proper submersion is a fiber bundle,  $P_\alpha|_{\tilde{\mathcal{U}}_\alpha \cap d_\alpha^{-1}([0, 2\nu])}$  is a fiber bundle.



Since the fibers of our local submersions are disks, by Equation (5.3.4), a fiber of  $P_\alpha|_{\tilde{\mathcal{U}}_\alpha \cap d_\alpha^{-1}([0, 2\nu])}$  is a disk. If

$$\mathcal{U}_\alpha \equiv \tilde{\mathcal{U}}_\alpha \cap d_\alpha^{-1}([0, 2\nu]),$$

then  $P_\alpha|_{\mathcal{U}_\alpha}$  is a fiber bundle with fiber  $\mathbb{R}^{n-l}$ , where  $l = \dim(S)$ . A priori, the structure group is  $\text{Diff}(\mathbb{R}^{n-l})$ , but  $\text{Diff}(\mathbb{R}^{n-l})$  deformation retracts to  $GL(n-l)$ . From this and Milnor's construction of classifying spaces ([17]), it follows that  $B\text{Diff}(\mathbb{R}^{n-l})$  deformation retracts to  $BGL(n-l)$ . So the structure group of  $P_\alpha|_{\mathcal{U}_\alpha}$  can be reduced to  $GL(n-l)$ , and  $P_\alpha|_{\mathcal{U}_\alpha}$  is a vector bundle.  $\square$

*Proof of Part 5 of the TNST.* Via an argument nearly identical to the proof of Proposition 6.2, we construct the submersions

$$Q^{S_j} : \mathcal{V}^{S_j} \setminus S_j \longrightarrow S_j.$$

To get Equation (0.0.5) we combine Part 5 of Theorem 2.14 and Corollary 5.5.  $\square$

*Proof of Part 1 of the TNST.* Part 1 of the TNST follows by combining the construction of the  $\mathcal{U}_\gamma^{S_i}$ s with the hypothesis that the elements of  $\mathcal{S}$  are pairwise disjoint and the fact that Theorem 2.14 holds for all sufficiently small  $\rho$ .  $\square$

**6.3. The Embeddings of the TNST.** Parts 3 and 4 of the TNST follow from the next result, wherein we construct the embedding  $\Phi_{\beta, \alpha} : G_\alpha \longrightarrow M_\beta$  of the Tubular Neighborhood Stability Theorem. The existence of an embedding  $G_\alpha \longrightarrow M_\beta$  is a consequence of Theorem 6.1 in [15]. To prove Part 4 of the TNST, we also need to show that  $\Phi_{\beta, \alpha}$  satisfies Equation (0.0.4). This is achieved via an appeal to Corollary 5.5.

**Proposition 6.4.** *Let  $X$  and  $\{M_\alpha\}_\alpha$  be as in the TNST.*

1. *The vector bundles*

$$P_\alpha^{S_i} : \mathcal{U}_\alpha^{S_i} \longrightarrow O_i \subset S_i,$$

*have euclidean metrics. Let  $\mathcal{U}_\alpha^i(r)$  be the bundle of open balls of radius  $r$  in  $\mathcal{U}_\alpha^i$ .*

2. *Set*

$$G_\alpha \equiv M_\alpha \setminus \cup_{i \in I} \mathcal{U}_\alpha^i(1).$$

*There is a  $C^1$ ,  $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ —embedding*

$$\Phi_{\beta, \alpha} : G_\alpha \longrightarrow M_\beta$$

*so that for all  $S_i \in \mathcal{S}$*

$$P_\alpha^{S_i} = P_\beta^{S_i} \circ \Phi_{\beta, \alpha},$$

*wherever both expressions are defined.*

*Proof.* Part 1 is true because every vector bundle over a paracompact space has a euclidean metric.

Using Part 5 of Theorem 2.14 and Corollary 5.5, we glue the embeddings  $\left(\mu_j^\beta\right)^{-1} \circ \mu_j^\alpha$  of Proposition 4.3 to get an immersion

$$\Phi_{\beta, \alpha} : G_\alpha \longrightarrow \Phi_{\beta, \alpha}(G_\alpha) \subset M_\beta$$

so that

$$P_\alpha^{S_i} = P_\beta^{S_i} \circ \Phi_{\beta,\alpha},$$

wherever both expressions are defined.

It follows from Inequalities (4.3.2) and (5.5.4) that  $\Phi_{\beta,\alpha}$  is also a  $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ -Hausdorff approximation. From Inequalities (4.3.1), (4.3.2), and (5.5.5) it follows that on  $B_j^\alpha(\rho)$

$$\left| \mu_j^\beta \circ \Phi_{\beta,\alpha} - \mu_j^\alpha \right|_{C^1} \leq \tau(\delta).$$

Combining this with the fact that  $\mu_j^\alpha$  and  $\mu_j^\beta$  are  $(\tau(\delta))$ -almost Riemannian embeddings, we see that  $\Phi_{\beta,\alpha}|_{B_j^\alpha(\rho)}$  is one-to-one. Because  $\Phi_{\beta,\alpha}$  is also a  $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ -Hausdorff approximation, it is one-to-one if  $\alpha$  and  $\beta$  are sufficiently large. Since  $\dim(M^\alpha) = \dim(M^\beta)$ ,  $\Phi_{\beta,\alpha}$  is an embedding.  $\square$

## 7. ESTABLISHING THEOREM B

In this section, we complete the proof of Theorem B by establishing Part 6 of the TNST and the Step 1 and 2 Schoenflies Lemmas. We prove Part 6 of the TNST in Subsection 7.1, and we establish the Step 1 and 2 Schoenflies Lemmas in Subsection 7.3.

**7.1. The Step 0 Schoenflies Lemma.** In this subsection, we prove Part 6 of the TNST.

Since  $\Phi_{\beta,\alpha}$  is a Gromov-Hausdorff approximation,

$$\Phi_{\beta,\alpha}(M_\alpha \setminus \{\cup_i \mathcal{U}_\alpha^i(3)\}) \subset M_\beta \setminus \{\cup_i \mathcal{U}_\beta^i(2)\} \cong M_\beta \setminus \{\cup_i \mathcal{U}_\beta^i(3)\}.$$

So it suffices to find a diffeomorphism of  $M_\beta$  that takes  $\Phi_{\beta,\alpha}(M_\alpha \setminus \{\cup_i \mathcal{U}_\alpha^i(3)\})$  to  $M_\beta \setminus \{\cup_i \mathcal{U}_\beta^i(2)\}$ . To accomplish this, it suffices to construct a vector field  $W$  that is non-vanishing on each

$$\mathcal{U}_\beta^S(10) \setminus \mathcal{U}_\beta^S(1),$$

transverse to the boundary of  $\Phi_{\beta,\alpha}(M_\alpha \setminus \{\cup_i \mathcal{U}_\alpha^i(3)\})$ , and transverse to the boundary of  $M_\beta \setminus \{\cup_i \mathcal{U}_\beta^i(2)\}$ . The existence of such a vector field follows from Part 1 of the next result, which completes the proof of Part 6 of the TNST.

**Proposition 7.2.** *For  $S \in \mathcal{S}$  and  $\gamma = \alpha$  or  $\beta$ , let*

$$P_\gamma^S : \mathcal{U}_\gamma^S \longrightarrow O_S \subset S$$

*be the vector bundle of Part 2 of the TNST.*

*1. There is a non-zero vector field  $W_\gamma^S$  on  $\mathcal{U}_\gamma^S(10) \setminus \mathcal{U}_\gamma^S(1)$  so that for all  $r \in [1, 10]$ ,  $W_\gamma^S$  is transverse to  $\partial \mathcal{U}_\gamma^S(r)$ , vertical for  $P_\gamma^S$ , and satisfies*

$$|W_\gamma^S| = 1 \text{ and } |D\Phi_{\beta,\alpha}(W_\alpha^S) - W_\beta^S| < \tau(\delta).$$

*2. Suppose in addition, that  $S \subset \bar{N} \setminus N$  for some  $N \in \mathcal{N}$ , and let*

$$P_\gamma^N : \mathcal{U}_\gamma^N \longrightarrow O_N \subset N$$

*be the vector bundle of Part 2 of the TNST. Then we can choose the  $W_\gamma^S$ s to also satisfy*

$$||DP_\alpha^N(W_\alpha^S)| - |W_\alpha^S|| < \tau(\varepsilon) \text{ and } |DP_\alpha^N(W_\alpha^S) - DP_\beta^N(W_\beta^S)| < \tau(\varepsilon). \quad (7.2.1)$$

*Proof.* Suppose  $S$  is in the closure of  $N$ , and let

$$p_k^{N,\gamma} : B_k^\gamma(3\rho_k^N) \longrightarrow \mathbb{R}^{\dim(N)}$$

and

$$p_j^{S,\gamma} : B_j^\gamma(\rho_j^S) \longrightarrow \mathbb{R}^{\dim(S)}$$

be the locally defined submersions pertaining to  $N$  and  $S$  from Theorem 3.4.

By Lemma 2.11 and Part 6 of Theorem 2.14, there is a unit vector field  $V_\gamma^S$  on  $\mathcal{U}_\gamma^S(10) \setminus \mathcal{U}_\gamma^S(1)$  so that

$$\left| Dp_j^{S,\gamma}(V_\gamma^S) \right| < \tau(\varepsilon) \quad (7.2.2)$$

$$\left| \left| Dp_k^{N,\gamma}(V_\gamma^S) \right| - |V_\gamma^S| \right| < \tau(\varepsilon), \text{ and} \quad (7.2.3)$$

$$\left| Dp_k^{N,\alpha}(V_\alpha^S) - Dp_k^{N,\beta}(V_\beta^S) \right| < \tau(\varepsilon). \quad (7.2.4)$$

Inequality (7.2.2) says that  $V_\gamma^S$  is almost vertical for all of the  $p_j^{S,\gamma}$ s. So it follows from Inequality (5.5.5) that  $V_\gamma^S$  is almost vertical for  $P_\gamma^S$ . Using the superscript  $\text{Vert}, P_\gamma^S$  to denote the vertical part with respect to  $P_\gamma^S$ , we set

$$W_\gamma^S \equiv \frac{(V_\gamma^S)^{\text{Vert}, P_\gamma^S}}{\left| (V_\gamma^S)^{\text{Vert}, P_\gamma^S} \right|}$$

and conclude that

$$\left| W_\gamma^S - (V_\gamma^S)^{\text{Vert}, P_\gamma^S} \right| < \tau(\varepsilon).$$

Inequality (7.2.1) follows from this and Inequalities (7.2.3), (7.2.4), and (5.5.5).

Since  $W_\gamma^S$  is vertical for  $P_\gamma^S$ , it is transverse to  $\partial\mathcal{U}_\gamma^S(r)$  for all  $r \in [1, 10]$ .

Via almost the same argument, we get

$$\left| D\Phi_{\beta,\alpha}(W_\alpha^S) - W_\beta^S \right| < \tau(\delta),$$

where  $\delta$  is as in the statement of the TNST. This completes the proofs of both parts.  $\square$

**7.3. The Schoenflies Lemmas.** We establish the Step 1 and 2 Schoenflies Lemmas by induction on the Ancestor Number of our strata. The induction is anchored at Step 0, where it is a consequence of Part 6 of the TNST.

The vector fields,  $D\Phi_{\beta,\alpha}^{a+1}(W_\alpha^S)$  and  $W_\beta^S$ , of Proposition 7.2 are the key to the induction step. Indeed, we write  $\mathfrak{U}_\gamma^a(r)$  for the union of all  $\mathcal{U}_\gamma^i(r)$ s for which the Ancestor Number of  $S_i$  is  $a$ , and we suppose that we have constructed smooth  $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ -embeddings

$$\begin{aligned} \Phi_{\beta,\alpha}^a & : M_\alpha \setminus \left\{ \cup_{i=a+1}^3 \mathfrak{U}_\gamma^i(3) \right\} \longrightarrow M_\beta \text{ and} \\ \Phi_{\beta,\alpha}^{a+1} & : M_\alpha \setminus \left\{ \cup_{i=a+2}^3 \mathfrak{U}_\gamma^i(3) \right\} \longrightarrow M_\beta \end{aligned}$$

so that

$$P_\alpha^k = P_\beta^k \circ \Phi_{\beta,\alpha}^a \text{ and } P_\alpha^k = P_\beta^k \circ \Phi_{\beta,\alpha}^{a+1}, \quad (7.3.5)$$

wherever all expressions are defined and so that the (Step  $a$ )–Schoenflies Lemma is satisfied. In other words,

$$\Phi_{\beta,\alpha}^a(M_\alpha \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\alpha^i(3)\}) = M_\beta \setminus \{\cup_{i=a+1}^3 \mathfrak{U}_\beta^i(3)\}.$$

It remains to derive the (Step  $a+1$ )–Schoenflies Lemma from the above data.

Since  $\Phi_{\beta,\alpha}^{a+1}$  is a Gromov-Hausdorff approximation,

$$\Phi_{\beta,\alpha}^{a+1}(M_\alpha \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\alpha^i(3)\}) \subset M_\beta \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\beta^i(2)\} \cong M_\beta \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\beta^i(3)\}.$$

So it suffices to find a diffeomorphism of  $M_\beta$  that takes  $\Phi_{\beta,\alpha}^{a+1}(M_\alpha \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\alpha^i(3)\})$  to  $M_\beta \setminus \{\cup_{i=a+2}^3 \mathfrak{U}_\beta^i(2)\}$ . To construct the diffeomorphism, we glue together the flows of the vector fields,  $D\Phi_{\beta,\alpha}^{a+1}(W_\alpha^S)$  and  $W_\beta^S$ , from Proposition 7.2.

This completes the proof of Theorem B, modulo the proofs of Theorem 5.3 and Corollary 5.5.

## 8. APPENDIX A: HOW TO GLUE $C^1$ –CLOSE SUBMERSIONS

In this section we prove Theorem 5.3 and Corollary 5.5. Before doing so we establish several inductive gluing tools in Subsubsection 8.1, and we prove a result about stability of intersection patterns in Subsubsection 8.6.

**8.1. Tools to Glue  $C^1$ –Close Submersions.** In this subsection we prove Key Lemma 8.5, the main inductive gluing lemma that will allow us to prove Theorem 5.3. First we establish several preliminary results.

**Lemma 8.2.** (*Submersion Isotopy Lemma*) *Let  $G \subset M$  be an open subset of a Riemannian  $n$ –manifold  $M$ . Let  $\pi : G \rightarrow \mathbb{R}^l$  be an  $\eta$ –almost Riemannian submersion, and let  $p : G \rightarrow \mathbb{R}^l$  be any submersion with*

$$|p - \pi|_{C^1} < \varepsilon.$$

*There are positive numbers  $\eta_1$  and  $\varepsilon_1$  that only depend on  $l$  so that if  $\eta \in (0, \eta_1)$  and  $\varepsilon \in (0, \varepsilon_1)$ , then the homotopy  $H : G \times [0, 1] \rightarrow \mathbb{R}^l$ ,*

$$H_t \equiv \pi + t(p - \pi),$$

*from  $p$  to  $\pi$  has the following properties.*

1.  $H_t$  is a submersion.
2.  $|H_t - \pi|_{C^1} < \varepsilon$  and  $|H_t - p|_{C^1} < \varepsilon$ .
3.  $|H_t - \pi|_{C^0} \leq |p - \pi|_{C^0}$  and  $|H_t - p|_{C^0} \leq |p - \pi|_{C^0}$ .
4. If  $Z \subset G$  is open and  $q : Z \rightarrow \mathbb{R}^l$  is a submersion with  $|q - \pi|_{C^1} < \varepsilon$  and  $|q - p|_{C^1} < \varepsilon$ , then  $|H_t - q|_{C^1} < \varepsilon$ .
5. If  $Z \subset G$  is open and  $q : Z \rightarrow \mathbb{R}^l$  is a submersion with  $|q - \pi|_{C^0} < \xi$  and  $|q - p|_{C^0} < \xi$ , then  $|H_t - q|_{C^0} < \xi$ .
6.  $|DH_{(x,t)}(0, \frac{\partial}{\partial t})| \leq |p - \pi|_{C^0}$ .
7. If  $F$  is a subset of  $G$  with  $p|_F = \pi|_F$ , then  $H_t|_F = p|_F = \pi|_F$  for all  $t$ .

*Proof.* There is an  $\varepsilon_{\text{Riem}} > 0$  so that for any Riemannian submersion  $\pi_{\text{Riem}} : G \rightarrow \mathbb{R}^l$ , any map  $h : G \rightarrow \mathbb{R}^l$  is a submersion, provided

$$|h - \pi_{\text{Riem}}|_{C^1} < \varepsilon_{\text{Riem}}.$$

Take  $\eta_1 = \varepsilon_1 = \frac{\varepsilon_{\text{Riem}}}{2}$ . Then any map  $h : G \rightarrow \mathbb{R}^l$  is a submersion provided

$$|h - \pi|_{C^1} < \varepsilon_1.$$

Since

$$H_t \equiv \pi + t(p - \pi),$$

Conclusions 2, 3, 4, and 5 follow from convexity of balls in Euclidean space, and Conclusion 7 follows from the definition of  $H$ . Conclusion 1 follows from Conclusion 2 and our choice of  $\varepsilon$ . Conclusion 6 follows from

$$DH_{(x,t)} \left( 0, \frac{\partial}{\partial t} \right) = p(x) - \pi(x).$$

□

**Lemma 8.3.** *For  $\zeta > 0$ , let  $W \subseteq V \subseteq G \subset M$  be three nonempty, open, pre-compact sets that satisfy*

$$\text{dist}(\overline{W}, \overline{G \setminus V}) > \zeta.$$

*There is a  $C^\infty$  function  $\omega : G \rightarrow [0, 1]$  that satisfies*

1.

$$\omega(x) = \begin{cases} 0 & \text{for } x \in W \\ 1 & \text{for } x \in G \setminus V \end{cases}$$

2.

$$|\nabla \omega| \leq \frac{2}{\zeta}.$$

*Proof.* Approximate  $\text{dist}(\overline{W}, \cdot)$  and  $\text{dist}(\overline{G \setminus V}, \cdot)$  by smooth functions in the  $C^0$ -topology. Choose sublevels  $C_1$  and  $C_2$  of these approximations so that  $W \subseteq C_1$ ,  $G \setminus V \subseteq C_2$ , and  $\text{dist}(C_1, C_2) > \zeta$ . Using the techniques of [9, 5], approximate  $\text{dist}(C_i, \cdot)$  by smooth functions  $f_{C_i}$  that satisfy  $f_{C_i} \geq 0$ ,  $|\nabla f_{C_i}| \leq 2$ , and  $f_{C_i}|_{C_i} \equiv 0$ . Since

$$\text{dist}(C_1, x) + \text{dist}(C_2, x) \geq \text{dist}(C_1, C_2) > \zeta,$$

and the technique of [9, 5] allows the approximation to be as close as we please in the  $C^0$ -topology, we can choose the  $f_{C_i}$ s so that they also satisfy

$$f_{C_1} + f_{C_2} > \zeta.$$

Then the function

$$\omega \equiv \frac{f_{C_1}}{f_{C_1} + f_{C_2}}$$

satisfies Property 1. Moreover,

$$\begin{aligned}
|\nabla\omega| &= \left| \frac{(f_{C_1} + f_{C_2}) \nabla f_{C_1} - f_{C_1} \nabla (f_{C_1} + f_{C_2})}{(f_{C_1} + f_{C_2})^2} \right| \\
&= \left| \frac{f_{C_2} \nabla f_{C_1} - f_{C_1} \nabla f_{C_2}}{(f_{C_1} + f_{C_2})^2} \right| \\
&\leq 2 \frac{f_{C_2} + f_{C_1}}{(f_{C_1} + f_{C_2})^2} \\
&\leq \frac{2}{\zeta},
\end{aligned}$$

as claimed.  $\square$

**Lemma 8.4.** (*Submersion Deformation Lemma*) *Let  $W \Subset V \Subset G \subset M$  satisfy the hypotheses of Lemma 8.3, and let  $\omega : G \rightarrow [0, 1]$  be as in the conclusion of Lemma 8.3. Let  $\pi : G \rightarrow \mathbb{R}^l$  be an  $\eta$ -almost Riemannian submersion, where  $\eta$  is as in Lemma 8.2. Let  $p : G \rightarrow \mathbb{R}^l$  be a submersion satisfying*

$$|p - \pi|_{C^1} < \varepsilon \text{ and } |p - \pi|_{C^0} < \xi < \varepsilon,$$

and let  $\varepsilon_1$  be as in Lemma 8.2.

If  $0 < \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta} < \varepsilon_1$ , then the map  $\psi : G \rightarrow \mathbb{R}^l$

$$\psi(x) = \pi(x) + \omega(x) \cdot (p - \pi)(x)$$

is a submersion with the following properties.

1.

$$\psi = \begin{cases} \pi & \text{on } W \\ p & \text{on } G \setminus V \end{cases}$$

2.

$$|\psi - \pi|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta} \text{ and } |\psi - p|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta}$$

3. If  $U \subset G$  is open and  $q : U \rightarrow \mathbb{R}^l$  is a submersion with  $|q - \pi|_{C^1} < \varepsilon$  and  $|q - p|_{C^1} < \varepsilon$ , then  $|\psi - q|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta}$ .

4. If  $U \subset G$  is open and  $q : U \rightarrow \mathbb{R}^l$  is a submersion with  $|q - \pi|_{C^0} < \xi$  and  $|q - p|_{C^0} < \xi$ , then  $|\psi - q|_{C^0} < \xi$ .

5. If  $F$  is a subset of  $G$  with  $p|_F = \pi|_F$ , then  $\psi|_F = p|_F = \pi|_F$ .

*Proof.* Part 1 is a consequence of the definitions of  $\psi$  and  $\omega$ .

Let  $H_t : G \rightarrow \mathbb{R}^l$  be the isotopy from Lemma 8.2. Since  $\psi(x) = H_{\omega(x)}(x)$ , Parts 4 and 5 follow from Parts 5 and 7 of Lemma 8.2.

For any  $x \in G$  and any  $v \in T_x M$ ,

$$D\psi_x(v) = D\pi_x(v) + \omega(x) D(p - \pi)_x(v) + \langle \nabla\omega, v \rangle (p - \pi)(x). \quad (8.4.1)$$

Since  $|p - \pi|_{C^1} < \varepsilon$ ,  $|\omega| \leq 1$ , and  $|\nabla \omega| \leq \frac{2}{\zeta}$ ,

$$\begin{aligned} |D\psi_x - D\pi_x| &\leq \varepsilon + |\nabla \omega| |p - \pi|_{C^0} \\ &\leq \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta}. \end{aligned}$$

By rewriting  $\psi$  as  $\psi = p + (1 - \omega) \cdot (\pi - p)$ , a similar argument gives

$$|D\psi_x - Dp_x| \leq \varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta}.$$

Combining the previous two displays gives us Part 2.

If  $q$  is as in Part 3, then by Part 4 of Lemma 8.2,

$$|(Dq)_x - (D\pi_x + \omega(x) D(p - \pi)_x)| < \varepsilon.$$

Combined with Equation (8.4.1) this gives us Part 3.

Combining Part 2 with our hypothesis that  $\varepsilon + \frac{2|p - \pi|_{C^0}}{\zeta} < \varepsilon_1$ , we see that  $\psi$  is a submersion.  $\square$

**Key Lemma 8.5.** *Let  $\tilde{M}$  and  $S$  be compact Riemannian manifolds. Let*

$$\begin{aligned} \tilde{W} \Subset \tilde{V} \Subset \tilde{G}, \tilde{O} &\subset \tilde{M} \text{ and} \\ G, O &\subset S \end{aligned}$$

*be pre-compact open sets with*

$$\text{dist}(\text{closure}(\tilde{W}), \text{closure}(\tilde{G} \setminus \tilde{V})) > \zeta.$$

*Let  $p_O : \tilde{O} \rightarrow O$ ,  $p_G : \tilde{G} \rightarrow G$  and  $\mu : G \rightarrow \mathbb{R}^l$  be  $\eta$ -almost Riemannian submersions with  $\mu$  a coordinate chart.*

*Suppose  $\tilde{W} \cap \tilde{O} \neq \emptyset$ ,  $p_G(\tilde{W}) \cap p_O(\tilde{O}) \neq \emptyset$ , and the restrictions of  $p_O$  and  $p_G$  to  $\tilde{O} \cap \tilde{G}$  satisfy*

$$|\mu \circ p_O - \mu \circ p_G|_{C^1} < \varepsilon \text{ and} \tag{8.5.1}$$

$$|\mu \circ p_O - \mu \circ p_G|_{C^0} < \xi < \varepsilon, \tag{8.5.2}$$

*where  $\varepsilon + \frac{2\xi}{\zeta} < \varepsilon_1$ , and  $\varepsilon_1$  is as in Lemma 8.2.*

*Then there is a submersion*

$$P : \tilde{W} \cup \tilde{O} \rightarrow P(\tilde{W} \cup \tilde{O}) \subset S$$

*so that*

$$P = \begin{cases} p_G & \text{on } \tilde{W} \\ p_O & \text{on } \tilde{O} \setminus \tilde{V}, \end{cases} \tag{8.5.3}$$

*and in addition, the following hold.*

1. *On  $\tilde{G} \cap \tilde{O}$ ,*

$$|\mu \circ P - \mu \circ p_G|_{C^1} < \varepsilon + \frac{2\xi}{\zeta} \text{ and } |\mu \circ P - \mu \circ p_O|_{C^1} < \varepsilon + \frac{2\xi}{\zeta}.$$

2. If  $\tilde{U} \subset \tilde{G} \cap \tilde{O}$  is open and  $q : \tilde{U} \rightarrow S$  is a submersion with  $|\mu \circ q - \mu \circ p_G|_{C^1} < \varepsilon$  and  $|\mu \circ q - \mu \circ p_O|_{C^1} < \varepsilon$ , then  $|\mu \circ P - \mu \circ q|_{C^1} < \varepsilon + \frac{2\xi}{\zeta}$ .
3. If  $\tilde{U} \subset \tilde{G} \cap \tilde{O}$  is open and  $q : \tilde{U} \rightarrow S$  is a submersion with  $|\mu \circ q - \mu \circ p_G|_{C^0} < \xi$  and  $|\mu \circ q - \mu \circ p_O|_{C^0} < \xi$ , then  $|\mu \circ P - \mu \circ q|_{C^0} < \xi$ .
4. If  $F$  is a subset of  $\tilde{O} \cap \tilde{G}$  with  $p_O|_F = p_G|_F$ , then  $P|_F = p_O|_F = p_G|_F$ .

*Proof.* By Lemma 8.4 there is a submersion  $\psi : \tilde{G} \cap \tilde{O} \rightarrow \psi(\tilde{G} \cap \tilde{O}) \subset \mathbb{R}^l$  so that

$$\psi = \begin{cases} \mu \circ p_G & \text{on } \tilde{W} \cap \tilde{O} \\ \mu \circ p_O & \text{on } (\tilde{G} \setminus \tilde{V}) \cap \tilde{O} \end{cases}$$

and

$$|\psi - \mu \circ p_G|_{C^1} < \varepsilon + \frac{2\xi}{\zeta} \text{ and } |\psi - \mu \circ p_O|_{C^1} < \varepsilon + \frac{2\xi}{\zeta}.$$

Therefore, the map

$$P : \tilde{W} \cup \tilde{O} \rightarrow S$$

defined by

$$P := \begin{cases} p_G & \text{on } \tilde{W} \\ \mu^{-1} \circ \psi & \text{on } \tilde{G} \cap \tilde{O} \\ p_O & \text{on } \tilde{O} \setminus \tilde{V} \end{cases}$$

is a well defined submersion satisfying Equation (8.5.3). Combining the definition of  $P$  with Parts 2, 3, 4, and 5 of the Submersion Deformation Lemma gives us Parts 1, 2, 3, and 4.  $\square$

## 8.6. Stability of Intersection Patterns.

**Proposition 8.7.** *Let  $\mathcal{C}$  be an ordered collection of  $m$  open subsets of a compact metric space  $X$ . Suppose that  $\mathcal{C}$  and  $\text{cl}(\mathcal{C})$  have the same intersection pattern. Let  $\mathcal{X}$  be the collection of compact subsets of  $X$  equipped with the Hausdorff metric, and let  $\mathcal{X}^m$  be the  $m$ -fold product of  $\mathcal{X}$ .*

*There is a neighborhood  $\mathcal{N}$  of  $\text{cl}(\mathcal{C})$  in  $\mathcal{X}^m$  with the following property: If  $\mathcal{D}$  is a collection of  $m$  open subsets of  $X$  with  $\text{cl}(\mathcal{D}) \in \mathcal{N}$ , then  $\mathcal{D}$  and  $\mathcal{C}$  have the same intersection pattern.*

*Proof.* Since  $\mathcal{C}$  and  $\text{cl}(\mathcal{C})$  have the same intersection pattern, there is an  $\varepsilon > 0$  so that if  $C_i, C_j \in \mathcal{C}$  are disjoint, then  $\text{dist}(c_i, c_j) > \varepsilon$  for all  $c_i \in C_i$  and  $c_j \in C_j$ . It follows that if  $D_i, D_j \in \mathcal{D}$  are close enough to  $C_i$  and  $C_j$ , then  $D_i$  and  $D_j$  are disjoint.

On the other hand, if  $x \in C_i \cap C_j$ , then there is an  $\eta > 0$  so that  $B(x, \eta) \subset C_i \cap C_j$ . It follows that  $D_i \cap D_j \neq \emptyset$  if the Hausdorff distances satisfy

$$\text{dist}_{\text{Haus}}(C_i, D_i) < \frac{\eta}{10} \text{ and } \text{dist}_{\text{Haus}}(C_j, D_j) < \frac{\eta}{10}.$$

$\square$

**Proposition 8.8.** *Adopt the hypotheses of Theorem 5.3, and let*

$$P_k : \cup_{i=1}^k \tilde{B}_i(\rho) \rightarrow P_k\left(\cup_{i=1}^k \tilde{B}_i(\rho)\right) \subset S$$



be a submersion with

$$\left| P_k - \mu_i^{-1} \circ \tilde{p}_i \Big|_{\tilde{B}_i(\rho)} \right|_{C^0} < \xi \quad (8.8.1)$$

for all  $i$ . If  $\xi$  is sufficiently small, then

$$P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) \cap B_{k+1}(\rho) \neq \emptyset$$

if and only if

$$\bigcup_{i=1}^k B_i(\rho) \cap B_{k+1}(\rho) \neq \emptyset.$$

*Proof.* We have  $P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) = \bigcup_{i=1}^k P_k \left( \tilde{B}_i(\rho) \right)$ , and Inequalities (8.8.1) and (5.3.1) give us that  $\bigcup_{i=1}^k P_k \left( \tilde{B}_i(\rho) \right)$  is Hausdorff close to  $\bigcup_{i=1}^k B_i(\rho)$ . So by Proposition 8.7,

$$\bigcup_{i=1}^k B_i(\rho) \cap B_{k+1}(\rho) \neq \emptyset$$

if and only if

$$P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) \cap B_{k+1}(\rho) \neq \emptyset.$$

□

### 8.9. Proofs of Theorem 5.3 and Corollary 5.5.

*Proof of Theorem 5.3.* Choose  $\varepsilon_0 > 0$  so that

$$\varepsilon_0 < \frac{\varepsilon_1}{2},$$

where  $\varepsilon_1$  is as in Lemma 8.2. Choose  $\xi_0, \eta > 0$  so that the conclusion of Proposition 8.8 holds with  $\xi = \xi_0$  and so that

$$(1 + \eta)^{2(\mathfrak{o}-1)} \varepsilon_0 + \frac{2}{\rho} \xi_0 (\mathfrak{o} - 1) (1 + \eta)^{2(\mathfrak{o}-1)} < \frac{\varepsilon_1}{2}.$$

Next we partition  $\left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l}$  into  $\mathfrak{o}$  subcollections of pairwise disjoint balls, where  $\mathfrak{o}$  is the order of  $\left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l}$ . To begin, we take  $\tilde{\mathcal{B}}_1(3\rho)$  to be a maximal subcollection of  $\left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l}$  that is pairwise disjoint, and in general, for  $j \in \{2, \dots, \mathfrak{o}\}$  we take  $\tilde{\mathcal{B}}_j(3\rho)$  to be a maximal pairwise disjoint subcollection of  $\left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l} \setminus \left\{ \tilde{\mathcal{B}}_1(3\rho) \cup \dots \cup \tilde{\mathcal{B}}_{j-1}(3\rho) \right\}$ . Then the collection  $\tilde{\mathcal{B}}_1(3\rho) \cup \dots \cup \tilde{\mathcal{B}}_{j+1}(3\rho)$  has order  $j$ , and  $\tilde{\mathcal{B}}_1(3\rho) \cup \dots \cup \tilde{\mathcal{B}}_{\mathfrak{o}}(3\rho) = \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_l}$ .

We let  $\tilde{\mathcal{B}}_j(\rho)$  be the  $\rho$ -balls that have the same centers as the  $\tilde{\mathcal{B}}_j(3\rho)$ s, and we let  $\mathcal{B}_j(3\rho)$  and  $\mathcal{B}_j(\rho)$  be the corresponding subcollections of  $\{B_j(3\rho)\}_{j=1}^{m_l}$  and  $\{B_j(\rho)\}_{j=1}^{m_l}$ . We use the superscript  $^u$  to denote the union of one of these subcollections. Thus for example,  $\tilde{\mathcal{B}}_1^u(3\rho)$  is the subset of  $M$  obtained by taking the union of each ball in  $\tilde{\mathcal{B}}_1(3\rho)$ .

For each  $j \in \{1, 2, \dots, \mathfrak{o}\}$  and each  $i$  with  $\tilde{B}_i(3\rho) \in \tilde{\mathcal{B}}_j(3\rho)$ , we let

$$\hat{p}_j : \tilde{\mathcal{B}}_j^u(3\rho) \longrightarrow \mathbb{R}^l$$

be given by

$$\hat{p}_j|_{B_i(3\rho)} = \tilde{p}_i,$$

and

$$\hat{\mu}_j : \mathcal{B}_j^u(3\rho) \longrightarrow \mathbb{R}^l$$

be given by

$$\hat{\mu}_j|_{B_i(3\rho)} = \mu_i.$$

The proof is by induction on the index  $j$  of the  $\tilde{\mathcal{B}}_j(3\rho)$ s. To formulate our induction statement for  $k \in \{1, \dots, \mathfrak{o}\}$ , we set

$$\mathcal{E}_k = (1 + \eta)^{2(k-1)} \varepsilon + \frac{2}{\rho} \xi (k-1) (1 + \eta)^{2(k-1)}. \quad (8.9.2)$$

Our  $k^{\text{th}}$  statement asserts the existence of a submersion

$$P_k : \cup_{j=1}^k \tilde{\mathcal{B}}_j^u(\rho) \longrightarrow P_k \left( \cup_{j=1}^k \tilde{\mathcal{B}}_j^u(\rho) \right) \subset S$$

so that for all  $s \in \{1, 2, \dots, \mathfrak{o}\}$  on  $\cup_{j=1}^k \tilde{\mathcal{B}}_j^u(\rho) \cap \tilde{\mathcal{B}}_s^u(3\rho)$ ,

$$|\hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s|_{C^0} < (1 + \eta)^{2k} \xi \text{ and} \quad (8.9.3)$$

$$|\hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s|_{C^1} < \mathcal{E}_k. \quad (8.9.4)$$

Setting  $P_1 = \hat{\mu}_1^{-1} \circ \hat{p}_1$  and appealing to Equations (5.3.2) and (5.3.3) anchors the induction.

Since the collection  $\left\{ \tilde{\mathcal{B}}_j^u(\rho) \right\}_{j=1}^{\mathfrak{o}}$  has order  $\mathfrak{o}$ ,  $\left( \cup_{j=1}^k \tilde{\mathcal{B}}_j^u(\rho) \right) \cap \tilde{\mathcal{B}}_{k+1}^u(\rho) \neq \emptyset$ . Combining this with  $\mathcal{E}_k < \mathcal{E}_{\mathfrak{o}} < \varepsilon_1$  allows us to apply Key Lemma 8.5 with  $p_O = P_k$  and  $p_G = \hat{\mu}_{k+1}^{-1} \circ \hat{p}_{k+1}$  to get a new submersion

$$P_{k+1} : \cup_{j=1}^{k+1} \tilde{\mathcal{B}}_j^u(\rho) \longrightarrow P_{k+1} \left( \cup_{j=1}^{k+1} \tilde{\mathcal{B}}_j^u(\rho) \right) \subset S.$$

It remains to verify hypotheses (8.9.3) $_{k+1}$  and (8.9.4) $_{k+1}$ . The induction hypothesis, (8.9.3) $_k$ , combined with our hypothesis that the differentials of the  $\hat{\mu}_i$ s are  $(1 + \eta)$ -bi-lipshitz gives

$$\begin{aligned} |\hat{\mu}_{k+1} \circ P_k - \hat{\mu}_{k+1} \circ \hat{\mu}_s^{-1} \circ \hat{p}_s|_{C^0} &= |(\hat{\mu}_{k+1} \circ \hat{\mu}_k^{-1}) \circ (\hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s)|_{C^0} \\ &< (1 + \eta)^2 (1 + \eta)^{2k} \xi \\ &= (1 + \eta)^{2(k+1)} \xi. \end{aligned}$$

So by Part 3 of the Key Lemma 8.5,

$$|\hat{\mu}_{k+1} \circ P_{k+1} - \hat{\mu}_{k+1} \circ (\hat{\mu}_s^{-1} \circ \hat{p}_s)|_{C^0} < (1 + \eta)^{2(k+1)} \xi,$$

and (8.9.3) $_{k+1}$  holds.

Combining (8.9.4) $_k$  with the fact that the differentials of the  $\hat{\mu}_i$ s are  $(1 + \eta)$ -bi-lipshitz gives

$$\begin{aligned} |\hat{\mu}_{k+1} \circ P_k - \hat{\mu}_{k+1} \circ \hat{\mu}_s^{-1} \circ \hat{p}_s|_{C^1} &= |\hat{\mu}_{k+1} \circ \hat{\mu}_k^{-1} \circ (\hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s)|_{C^1} \\ &< (1 + \eta)^2 (\mathcal{E}_k). \end{aligned}$$

So by Part 2 of Key Lemma 8.5 and (8.9.3)<sub>k</sub>,

$$\begin{aligned}
 \left| \hat{\mu}_{k+1} \circ \hat{P}_{k+1} - \hat{\mu}_{k+1} \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right|_{C^1} &< (1+\eta)^2 (\mathcal{E}_k) + \frac{2}{\rho} (1+\eta)^{2k} \xi \\
 &= (1+\eta)^2 \left( (1+\eta)^{2(k-1)} \varepsilon + \frac{2}{\rho} \xi (k-1) (1+\eta)^{2(k-1)} \right) \\
 &\quad + \frac{2}{\rho} (1+\eta)^{2k} \xi \\
 &= (1+\eta)^{2k} \varepsilon + \frac{2}{\rho} \xi k (1+\eta)^{2k} \\
 &= \mathcal{E}_{k+1}.
 \end{aligned}$$

To complete the proof, we need to establish Equation (5.3.4). To do so, we re-index so that  $\tilde{B}_{m_l}(\rho) \subset \tilde{\mathcal{B}}_{\mathfrak{o}}^u(3\rho)$  and notice that

$$P|_{\tilde{\mathcal{B}}_{\mathfrak{o}}^u(\rho)} = P_{\mathfrak{o}}|_{\tilde{B}_{m_l}(\rho)} = \hat{\mu}_{\mathfrak{o}}^{-1} \circ \hat{p}_{\mathfrak{o}}$$

by Equation (8.5.3). □

*Proof of Corollary 5.5.* First apply Theorem 5.3 to construct a submersion  $\tilde{P} : \cup_{i=1}^{m_l} \tilde{B}_i(\rho) \rightarrow S$  that is close to the  $\tilde{p}_i$ s in the sense that Inequalities (5.3.5) and (5.3.6) hold.

Since the order of  $\{B_i(3\rho_R)\}_{i \in I_R}$  is  $\mathfrak{o}$ , as in the proof of Theorem 5.3, for each  $j \in \{1, 2, \dots, \mathfrak{o}\}$ , we construct a subcollection  $\mathcal{B}_j(3\rho_R)$  of  $\{B_i(3\rho_R)\}_{i \in I_R}$  so that the balls of  $\mathcal{B}_j(3\rho_R)$  are pairwise disjoint, and the collection  $\mathcal{B}_1(3\rho_R) \cup \dots \cup \mathcal{B}_j(3\rho_R)$  has order  $j$ .

For each  $j \in \{1, 2, \dots, \mathfrak{o}\}$ , we set

$$p_j \equiv \mu_j \circ Q \circ R : \mathcal{B}_j^u(3\rho_R) \rightarrow \mathbb{R}^l,$$

and note that since the  $p_j$ s are all coordinate representations of the same submersion,  $Q \circ R$ ,

$$\text{Inequalities (5.3.2) and (5.3.3) hold with } \xi = \varepsilon = 0 \tag{8.9.5}$$

and the  $p_i$ s playing the role of the  $\tilde{p}_i$ s. Using this, for each  $j \in \{1, 2, \dots, \mathfrak{o}\}$ , we successively apply the proof of Theorem 5.3 to deform  $\tilde{P}$  on each  $\mathcal{B}_j(3\rho_R)$  so that it ultimately equals  $Q \circ R$  on  $\cup_{j=1}^{\mathfrak{o}} \mathcal{B}_j^u(\rho_R)$ . For the first deformation, this is possible because Inequalities (5.5.1), (5.5.2), and (8.9.5) tell us that the  $p_j$ s are close to the  $\tilde{p}_j$ s. Via (5.3.5) and (5.3.6) it follows that the  $p_j$ s are close to local representations of  $\tilde{P}$ . In other words, we have that Inequalities (8.5.1) and (8.5.2) hold with  $p_O = \tilde{P}$  and  $p_G = Q \circ R$ . This continues to be possible for subsequent deformations because Parts 2 and 3 of Key Lemma 8.5 tell us our deformations preserve Inequalities (8.5.1) and (8.5.2), provided  $\xi$  and  $\varepsilon$  are sufficiently small.

To explain why  $P = Q \circ R$  on  $\cup_{j=1}^{\mathfrak{o}} \mathcal{B}_j^u(\rho_R)$ , we let  $\tilde{P}_0, \tilde{P}_1, \dots, \tilde{P}_{\mathfrak{o}}$  be the deformations of  $\tilde{P} = \tilde{P}_0$ . By combining Equation (8.5.3) with the fact that  $p_1 = \mu_1 \circ Q \circ R$ , it follows that

$$\tilde{P}_1 \equiv Q \circ R$$

on  $\mathcal{B}_1^u(\rho_R)$ . By the same reasoning, we have

$$\tilde{P}_k \equiv Q \circ R$$

on  $\mathcal{B}_k^u(\rho_R)$ , and Part 4 of Lemma 8.5 gives, via induction, that after the  $k^{th}$  deformation, we have

$$\tilde{P}_k \equiv Q \circ R$$

on  $\cup_{j=1}^k \mathcal{B}_j^u(\rho_R)$ . So setting  $P \equiv \tilde{P}_o$ , we see that  $P = Q \circ R$  on  $\cup_{j=1}^o \mathcal{B}_j^u(\rho_R)$ .  $\square$

## 9. APPENDIX B: CONVENTIONS AND NOTATIONS

We assume throughout that all metric spaces are complete, and the reader has a basic familiarity with Alexandrov spaces, including but not limited to the seminal paper by Burago, Gromov, and Perelman ([1]). Let  $X$ ,  $\mathcal{S} = \{S_i\}_{i \in I}$ ,  $\mathcal{N}$ , and  $\mathcal{K}$  be as in Theorem B, and let  $p, x$ , and  $y$  be points of  $X$ .

1. We call minimal geodesics in  $X$  *segments*.
2. We denote comparison angles with  $\tilde{\angle}$ .
3. We let  $\Sigma_p X$  and  $T_p X$  denote the space of directions and tangent cone at  $p$ , respectively, and we let  $*$  denote the cone point.
4. For a geodesic direction  $v \in T_p X$ , we let  $\gamma_v$  be the segment whose initial direction is  $v$ .
5. Following [22], given a subset  $A \subset X$ ,  $\uparrow_x^A \subset \Sigma_x$  denotes the set of directions of segments from  $x$  to  $A$ , and  $\uparrow_x^A \in \uparrow_x^A$  denotes the direction of a single segment from  $x$  to  $A$ . For  $x \in S_i \subset X$  and  $A \subset S_i$ , we write  $(\uparrow_x^A)_{S_i}$  or  $(\uparrow_x^A)_{S_i}$  if we are referring to intrinsic segments of  $S$  and  $(\uparrow_x^A)_X$  or  $(\uparrow_x^A)_X$  if we are referring to extrinsic segments of  $X$ .
6. For a differentiable map  $\Phi$  we write  $D\Phi$  for the differential of  $\Phi$ . If  $\Phi$  is real valued, we write  $D_v(\Phi)$  for the derivative of  $\Phi$  in the  $v$  direction.
7. Given a subset  $A \subset X$ , we say that  $\text{dist}_A(\cdot)$  is  $(1 - \varepsilon)$ -regular at  $x$  if there is a  $v \in \Sigma_x$  so that the derivative of  $\text{dist}_A(\cdot)$  in the direction  $v$  satisfies

$$D_v \text{dist}_A > 1 - \varepsilon.$$

8. We let  $px$  denote a segment from  $p$  to  $x$ .
9. We let  $\angle(x, p, y)$  denote the angle of a hinge formed by segments  $px$  and  $py$  and  $\tilde{\angle}(x, p, y)$  denote the corresponding comparison angle.
10. Following [19], we let  $\tau : \mathbb{R}^k \rightarrow \mathbb{R}_+$  be any function that satisfies

$$\lim_{x_1, \dots, x_k \rightarrow 0} \tau(x_1, \dots, x_k) = 0,$$

and, abusing notation, we let  $\tau : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}$  be any function that satisfies

$$\lim_{x_1, \dots, x_k \rightarrow 0} \tau(x_1, \dots, x_k | y_1, \dots, y_n) = 0,$$

provided  $y_1, \dots, y_n$  remain fixed. When making an estimate with a function  $\tau$ , we implicitly assert the existence of such a function for which the estimate holds.  $\tau$  often depends on the limit space  $X$  and/or its dimension, but we make no other mention of this.

11. We identify  $\mathbb{R}^l$  with  $\mathbb{R}^l \times \{0\}$ , and we let  $\pi_l : \mathbb{R}^l \times \mathbb{R}^{n-l} \rightarrow \mathbb{R}^l$  be orthogonal projection to the first  $l$  factors of  $\mathbb{R}^n$ .
12. For  $\lambda \in \mathbb{R}$ , we call a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  (strictly)  $\lambda$ -concave if and only if the function  $g(t) = f(t) - \lambda t^2/2$  is (strictly) concave.

13. If  $U$  is an open subset of an Alexandrov space  $X$ , we call  $f : U \rightarrow \mathbb{R}$ , (strictly)  $\lambda$ -concave if and only if its restriction to every geodesic is (strictly)  $\lambda$ -concave.
14. We abbreviate the statement “ $\{M_\alpha\}_{\alpha=1}^\infty$  converges to  $X$  in the Gromov–Hausdorff topology” with the symbols,  $M_\alpha \xrightarrow{GH} X$ .
15. Let  $V$  and  $W$  be normed vector spaces. For a linear map  $L : V \rightarrow W$ , we set  $|L| = \max \left\{ \left| L \left( \frac{v}{|v|} \right) \right| \mid v \in V \setminus \{0\} \right\}$ .
16. Let  $U \subset M$  be open and  $\Phi : U \rightarrow \mathbb{R}^n$  be  $C^1$ . We write

$$|\Phi|_{C^0} \equiv \sup_{x \in U} \{|\Phi(x)|\} \text{ and}$$

$$|\Phi|_{C^1} \equiv \max \left\{ |\Phi|_{C^0}, \sup_{x \in U} \{|D\Phi_x|\} \right\}$$

17. We call a submersion,  $\pi$ ,  $\eta$ -almost Riemannian if and only if for all unit horizontal vectors,
- $$|D\pi(v) - 1| < \eta.$$
18. An  $\eta$ -embedding ( $\eta$ -homeomorphism) is an embedding (homeomorphism) that is also an  $\eta$ -Gromov-Hausdorff approximation.
  19. Volume of subsets of Alexandrov spaces means rough volume as defined in [1].
  20. For  $\lambda > 0$ , we write

$$\lambda X$$

for the metric spaces obtained from  $X$  by rescaling all distances by  $\lambda$ .

21. We write  $N$  or  $N_i$  for an element of  $\mathcal{N}$ ;  $K$  or  $K_i$  for an element of  $\mathcal{K}$ ; and  $S$  or  $S_i$  for an element of  $\mathcal{S}$ . Thus we redundantly write

$$\begin{aligned} \mathcal{S} &= \{S_i\}_i \\ &= \{\mathcal{K}_k\}_k \cup \{N_n\}_n. \end{aligned}$$

22. We set

$$\mathcal{S}^{\text{ext}} \equiv \mathcal{S} \cup (X \setminus \cup_{S \in \mathcal{S}} S).$$

23. We use superscripts to denote components of vectors in subspaces. So, for example, if  $V$  is a subspace of  $W$ , then  $U^V$  is the component of  $U$  in  $V$ .
24. We write  $\mathbb{S}^n$  for the unit sphere in  $\mathbb{R}^{n+1}$ .
25. We set

$$B(p, r) \equiv \{x \in X \mid \text{dist}(x, p) < r\}.$$

26. We use  $A \Subset B$  to mean that the closure of  $A$  is contained in the interior of  $B$ .
27. A collection  $\{C_i\}_{i \in I}$  has order  $\mathfrak{o}$  if some  $x$  is in  $\mathfrak{o}$  of the  $C_i$ s and no  $x$  is in  $(\mathfrak{o}+1)$  of the  $\{C_i\}_{i \in I}$ s.

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